The Steiner Ratio for Obstacle-Avoiding Rectilinear Steiner Trees

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Abstract

We consider the problem of finding a shortest rectilinear Steiner tree for a given set of points in the plane in the presence of rectilinear obstacles that must be avoided. We extend the *Steiner ratio* to the obstacle-avoiding case and show that it is equal to the Steiner ratio for the obstacle-free case.

1 Introduction

Given a set of points (also called *terminals*) and a set of obstacles in the plane, an obstacle-avoiding rectilinear Steiner minimum tree (OAR-SMT) is a tree of shortest length, composed solely of vertical and horizontal line segments, connecting the points and avoiding the interior of the obstacles. The OAR-SMT problem has important applications in VLSI design. For extensive surveys of Steiner tree problems, refer to [5] and [7].

A Steiner tree may contain vertices different from the points to be connected, namely Steiner points. If we do not allow Steiner points, then the problem becomes the minimum spanning tree problem. Whereas the Steiner problem has been proven to be NP-hard in both Euclidean and rectilinear metrics [3, 2], it is easy to determine the minimum spanning tree. Consequently, we are interested in the quality of a minimum spanning tree as an approximation to the minimum Steiner tree for various versions of the problem. The *Steiner ratio* is defined to be the maximum, over all instances, of the ratio of the length of a minimum spanning tree to the length of a Steiner minimum tree (SMT). For every metric space, the Steiner ratio is between 1 and 2 [4]. For the Euclidean Steiner tree problem (obstacle-free case), the Steiner ratio is $\frac{2}{\sqrt{3}}$. This was conjectured by Gilbert and Pollack in 1966 [4] and was proved in 1992 by Du and Hwang [1]. For the rectilinear Steiner tree problem, Hwang [6] proved earlier that the Steiner ratio is $\frac{3}{2}$.

What is the most natural generalization of this to the case of obstacles? An edge of a spanning tree must walk around the obstacles in this case. It seems natural to allow Steiner points at corners of obstacles. This does not lead to a polynomially solvable problem but, as we show here, does lead to interesting Steiner ratio results. We call an Steiner point *anchored* if it is at the corner of an obstacle. An *anchored-OAR-SMT* is then defined as an obstacle-avoiding rectilinear SMT in which all Steiner points are anchored. Note that when there are no obstacles, an anchored-OAR-SMT is a minimum spanning tree. We define the obstacle-avoiding Steiner ratio as the worst case ratio of the length of an anchored-OAR-SMT to the length of an OAR-SMT. We show that this ratio is $\frac{3}{2}$, which is the same as the Steiner ratio for the obstacle-free case.

Note that if we do not allow Steiner points, we do not get an interesting ratio in the case of obstacles. The worst case ratio between the lengths of an obstacle-avoiding minimum spanning tree and a Steiner minimum tree is 2, which is equal to the Steiner ratio in a generic metric space.

In the remainder of this paper, we use Steiner tree to mean a rectilinear obstacle-avoiding Steiner tree, unless otherwise specified.

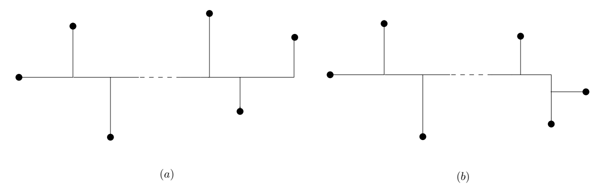

Figure 1: Two canonical forms of full Steiner trees.

2 Canonical Trees

In this section we show that it suffices to prove our Steiner ratio result for *canonical* Steiner trees, which have the forms shown in Figure 1. This was proved for the case without obstacles by Hwang [6] and we follow his approach. A *canonical* Steiner tree is defined as follows:

Definition 1 (Canonical Trees) *A rectilinear minimum Steiner tree is* canonical *if it has one of the following forms, possibly after a rotation:*

i. *All Steiner points and the leftmost terminal lie on a horizontal line. All Steiner points are connected to exactly one terminal by a vertical edge. These vertical edges alternatingly extend up and down. The rightmost and leftmost Steiner points are connected to a second terminal by a horizontal edge or a corner (Figure 1(a)).*

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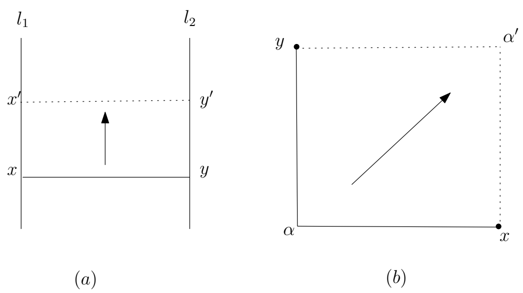

Figure 2: (a) Shifting. (b) Flipping.

ii. As above except that the two rightmost Steiner points are connected together by a corner (Figure 1(b)).

Hwang's first step is to reduce to *full* Steiner trees, which are Steiner trees in which terminals appear only as leaves. To justify this, note that we can cut a Steiner tree at any non-leaf terminal to obtain full Steiner subtrees. Similarly, we can cut an obstacle-avoiding Steiner tree at non-leaf terminals and at obstacle corners to obtain *full* obstacle-avoiding Steiner trees, in which terminals and obstacle corners appear only as leaves. It is sufficient to prove the ratio result for full subtrees (this is justified more formally below). Note that obstacle corners are then regarded as terminals in the full subtrees.

Hwang's next step is to apply *shifting* and *flipping* operations (Figure 2) to transform any minimum Steiner tree to one whose full subtrees are canonical. In a shift, a segment xy incident to two parallel lines l_1 and l_2 and containing no terminals or Steiner points other than possibly x and y is replaced by segment $x'y'$ also incident to l_1 and l_2 and parallel to xy. In a flip, two segments $x\alpha$ and $y\alpha$ meeting at the corner α and containing no terminals or Steiner points other than possibly x and y are replaced by segments $x\alpha'$ and $y\alpha'$, such that $x\alpha y\alpha'$ is a rectangle. These operations do not increase the length of a Steiner tree, and therefore map an SMT to another SMT. Two Steiner trees are said to be *equivalent* if one can be transformed to the other by shifting and flipping. To deal with obstacles, we first perform shifts and flips to bump into as many obstacles as possible. We then claim that further shifts and flips, as performed in Hwang's reduction, can ignore the obstacles. More formally:

Lemma 1 *Let T be an OAR-SMT such that, among all equivalent OAR-SMTs, T has the maximum number of full subtrees. Then shift and flip transformations can be done on T as if there were no obstacles, without violating the obstacle-avoiding property.*

Proof. Assume by contradiction that there exists a transformation H mapping T to T' such that T' violates the obstacle-avoiding property. Let H be a shift from segment xy to $x'y'$ such that $x'y'$ intersects an obstacle (Figure 3). Then, there must exist at least one obstacle corner inside the rectangle $xyy'x'$. Let c be the closest such obstacle corner to xy. We can shift xy to c to increase the number of full subtrees by at least one, thus contradicting the assumption that T has the maximum number of full subtrees. Next, let H be a flip from the corner $x\alpha y$ to the corner $x\alpha' y$ such that either $x\alpha'$ or $y\alpha'$ intersects with an obstacle. Since the obstacles are rectilinear, there must exist an obstacle corner inside the rectangle $x\alpha y\alpha'$. Let c be the closest such obstacle to α. We can flip α to c and increase the number of full subtrees by at least one, again leading to a contradiction.

□

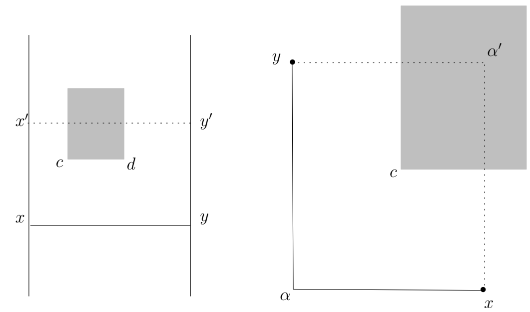

Figure 3: Shifting and flipping in the presence of obstacles.

Starting with an OAR-SMT with the maximum number of full subtrees, we can therefore apply Hwang's proof steps and reductions to get a canonical OAR-SMT:

Theorem 1 *For any terminal set P and obstacle set O there is an OAR-SMT whose full subtrees are in canonical form.*

Notation We call the horizontal line connecting the Steiner points the *spine* and denote it by E. We call the vertical lines connecting the terminals to the spine the *ribs*. Let n_u and n_l be the number of the ribs above and below the spine, respectively ($n_l = n_u$ or $n_l = n_u \pm 1$). Let $R_1, \ldots, R_{n_u}$ and $r_1, \ldots, r_{n_l}$ denote the ribs above and below the spine, in the order of x coordinate, respectively. Let T_i and t_i denote the terminals located on R_i and r_i, respectively. Denote by S_i and s_i the Steiner points at which R_i and r_i meet the spine, respectively. The rightmost rib and the leftmost rib (of length zero) meet the the spine at a corner point or a terminal, which for convenience of notation, we also denote by S_i and s_i for $i = 1, n_l, n_u$. We define a *pocket* as a subtree connecting three terminals consecutive in x-ordering. Without loss of generality, we assume that

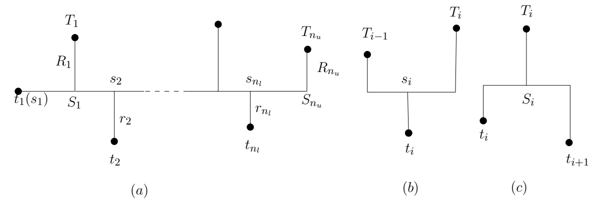

Figure 4: (a) A canonical Steiner tree. (b) Upper pocket P_i. (c) Lower pocket p_i.

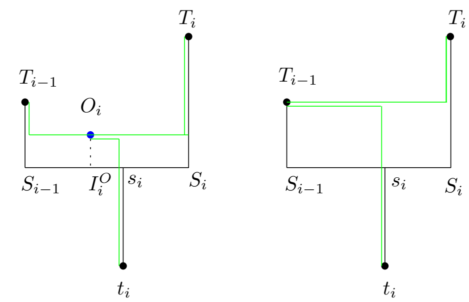

Figure 5: The Green Tree.

the second leftmost rib is an upper rib. Then the ith upper pocket, denoted by P_i, connects T_{i-1}, t_i and T_i via s_i and the ith lower pocket, denoted by p_i, connects t_i, T_i and t_{i+1} via S_i. These notations are illustrated in Figure 4.

3 Rectilinear Steiner Ratio With Obstacles

Let T^* and AT^* denote an OAR-SMT and an anchored-OAR-SMT for a terminal set P, respectively.

Theorem 2

$$\frac{|AT^*|}{|T^*|} \leq \frac{3}{2}$$

Proof. The proof only needs to be established for canonical Steiner trees. To see why, consider an OAR-SMT T' equivalent to T^* whose full subtrees are canonical. Such a tree exists by Theorem 1. Assume that Theorem 2 holds for each full subtree F_i. Therefore, there exists an anchored OAR-SMT G_i for the terminal set spanned by F_i such that $|G_i| \leq \frac{3}{2}|F_i|$. The union of all G_i's, denoted by G, is an anchored Steiner tree spanning P that is no longer than $\frac{3}{2}$ times the length of T^*.

We therefore assume that T^* is a canonical Steiner minimal tree. We will build a pair of anchored Steiner trees on P whose lengths add up to $3|T^*|$. The smaller tree will therefore have length at most $\frac{3}{2}|T^*|$.

We use the notation given at the end of Section 2. First, we assume that T^* is a type (i) canonical tree. We identify a subset of obstacle corners, called *critical corners*, and use them as Steiner points in the construction of the two trees. All other obstacle corners can be ignored.

For each upper (lower) pocket consider the set of all obstacle corners located above (below) the spine and between the two upper (lower) ribs. The height of such an obstacle corner is defined as its distance from the spine. We can restrict attention to the obstacles whose heights are less than the length of the shorter rib of their pocket. We define a *U-critical (L-critical) corner* as the obstacle corner with the minimum height in this set, breaking ties arbitrarily. If there is no such obstacle corner, the critical corner is the terminal located on the shorter upper (lower) rib and we refer to it as a *virtual* critical corner. Note that the length of the shortest path between a critical corner and a terminal in its pocket is equal to their rectilinear distance. Let O_i (o_i) denote the U-critical (L-critical) corner in the upper pocket P_i (lower pocket p_i), and let I_i^O (I_i^o) be its image projected on the spine. Let H_i^O (H_i^o) be the height of O_i (o_i).

3.1 The green tree

For each upper pocket, we connect the three terminals t_i, T_{i-1} and T_i to the U-critical corner O_i. See Figure 5. The length of the subtree is:

$$|T_{P_i}| = |R_{i-1}| + |R_i| - |H_i^o| + |r_i| + |S_{i-1}S_i| + |s_i I_i^O|$$

We connect the boundary terminals, if not included in any upper pocket, directly to the next terminal in the x-ordering. Summing over all upper pockets, the length of the green tree is:

$$|T_{green}| = 2\sum_{i=1}^{n_u} |R_i| + \sum_{i=1}^{n_l} |r_i| - \sum_{i=1}^{n_u} |H_i^O| + |E| + \sum_{i=1}^{n_u} |s_i I_i^O|$$

3.2 The red tree

A U-critical corner O_j is *involved* in a lower pocket p_i, if O_j's image on the spine, I_j^O, is located between the boundary Steiner points s_i and s_{i+1}. O_j can be either involved in the pocket p_j or p_{j-1}.

For each lower pocket p_i, there can be 0, 1 or 2 U-critical corners involved in the pocket. We consider three cases:

Case 1: There is one U-critical corner, O_j ($j = i$ or $i+1$), involved in p_i. We connect t_i, T_{i-1} and T_i to O_j. See Figure 6 (a) and (b). In this case, the length of the subtree is:

$$\begin{aligned} |T_{p_i}| &= |R_i| + |r_i| + |r_{i+1}| + |H_i^O| + |s_i s_{i+1}| + |S_i I_j^O| \\ &\leq |R_i| + |r_i| + |r_{i+1}| + |H_i^O| + 2|s_i s_{i+1}| - |s_j I_j^O| \end{aligned}$$

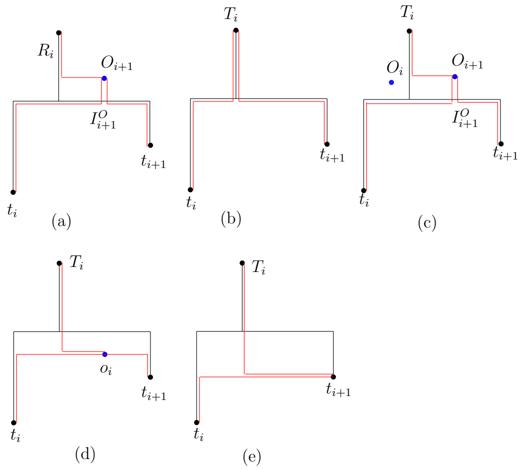

Figure 6: The Red Tree: (a) one (non-virtual) upper critical corner. (b)Virtual upper critical corner. (c)Two upper critical corners. (d) No upper critical corner, at least one (non-virtual) lower critical corner. (e) No upper critical corner, virtual or no lower critical obstacle.

Case 2: There are two U-critical corners, O_i and O_{i+1}, involved in p_i. We connect t_i, T_{i-1} and T_i to O_{i+1}. See Figure 6(c). The length of the subtree is:

$$|T_{p_i}| \quad = \quad |R_i| + |r_i| + |r_{i+1}| + |H^O_{i+1}| + |s_i s_{i+1}| + |S_i I^O_{i+1}|$$

We want the term $|H^O_i| - |s_i I^O_i|$ to appear exactly once for each U-critical corner O_i, so that it is canceled by the term $|s_i I^O_i| - |H^O_i|$ in the green tree's length. Therefore, we re-write the length of the subtree as:

$$\begin{aligned} |T_{p_i}| \quad &\leq \quad |R_i| + |r_i| + |r_{i+1}| + |H^O_i| + |H^O_{i+1}| \\ &+ \quad 2|s_i s_{i+1}| - |s_i I^O_i| - |s_{i+1} I^O_{i+1}| \end{aligned}$$

Case 3: There are no U-critical corners involved in p_i. In this case, we use the L-critical corner in the pocket, o_i, as a Steiner point and connect the three terminals t_i, T_{i-1} and T_i to o_i. See Figure 6 (d) and (e).

The length of the subtree is:

$$|T_{p_i}| \leq |R_i| + |r_i| + |r_{i+1}| + 2|s_i s_{i+1}|$$

We connect the boundary terminals, if not included in any lower pocket, directly to the next terminal in x-ordering. Summing over all lower pockets, the length of the red tree is :

$$|T_{red}| \leq \sum_{i=1}^{n_u} |R_i| + 2\sum_{i=1}^{n_l} |r_i| + \sum_{i=1}^{n_u} |H^O_i| + 2|E| - \sum_{i=1}^{n_u} |s_i I^O_i|$$

Now we add up the lengths of the two trees together:

$$|T_{green}| + |T_{red}| \leq 3\sum_{i=1}^{n_u} R_i + 3\sum_{i=1}^{n_l} r_i + 3|E| = 3|T^*|$$

This proves that the length of the shorter tree is at most $\frac{3}{2}$ times the length of T^*.

Now assume that T^* is a type (ii) canonical tree. First, we ignore the exceptional terminal, call it u, and build the red and green trees as above. Then, we modify the red tree so that the path between t_{n_l} and T_{n_u} passes through u, and in the green tree, we connect u to T_{n_u}. It is easy to see that the lengths of the two trees still add up to less than 3 times the length of T^*.

Finally, the $\frac{3}{2}$ bound for the obstacle-avoiding Steiner ratio is tight, since it clearly cannot be less than the Steiner ratio for the obstacle-free case.

□

4 Future work

We are working on an approximation algorithm to compute the anchored-OAR-SMT. We conjecture that in the Euclidean case, the obstacle-avoiding Steiner ratio is $\frac{2}{\sqrt{3}}$, the same as the Steiner ratio for the Euclidean obstacle-free case.

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