On a Cone Covering Problem

Khaled Elbassioni*
elbassio@mpi-sb.mpg.de

Hans Raj Tiwary†
hansraj@cs.uni-sb.de

Abstract

Given a set of polyhedral cones $\mathcal{C}_1, \cdots, \mathcal{C}_k \subset \mathbb{R}^d$, and a convex set D, does the union of these cones cover the set D? In this paper we consider the computational complexity of this problem for various cases such as whether the cones are defined by extreme rays or facets, and whether D is entire $\mathbb{R}^d$ or an affine subspace $\mathbb{R}^t$. As a consequence, we show that the problem of checking if the union of a given set of convex polytopes is convex is coNP-complete, thus answering a question of Bemporad et al. [3].

1 Introduction

Let $S \subseteq \mathbb{R}^d$ be a set of points in $\mathbb{R}^d$. The *conic hull* of S, denoted by cone(S), is the set of all non-negative linear combinations of points in S, i.e., cone$(S) = \{\sum_{p\in S} \mu_p p : \mu_p \geq 0 \text{ for all } p \in S\}$. It is well-known that any polyhedral cone cone(S) can be written equivalently as the intersection of finitely many half-spaces, i.e., cone$(S) = \{x \in \mathbb{R}^d : Ax \leq 0\}$, where $A \in \mathbb{R}^{m\times d}$. The two representations are called the $\mathcal{V}$- and the $\mathcal{H}$-representations, respectively.

In this note we are interested in the complexity of covering problems of the following form:

ConeCover$(\mathcal{C}, D)$: Given a collection of cones $\mathcal{C} = C_1, \ldots, C_k$, and a convex set D, does $\bigcup_{i=1}^k C_i \not\supseteq D$?

A polytope P is the convex hull of a finite set S of points in $\mathbb{R}^d$, and it can also be written in one of two equivalent forms: $P = \text{conv}(S) = \{\sum_{p\in S} \mu_p p : \mu_p \geq 0 \text{ for all } p \in S, \sum_{p\in S} \mu_p = 1\}$ ($\mathcal{V}$-representation), or $P = \{x \in \mathbb{R}^d | Ax \leq \mathbf{1}\}$, where $\mathbf{1}$ is the vector in which each component is 1 ($\mathcal{H}$-representation)[1]. A polyhedron Q is the Minkowski sum of a polytope P and a cone C: $Q = P + C \stackrel{\text{def}}{=} \{x + y | x \in P, \text{ and } y \in C\}$. Similarly, one can also consider the problem PolytopeCover$(\mathcal{P}, D)$: Given a collection of polytopes $\mathcal{P} = P_1, \ldots, P_k$, and a convex polytope D, does $\bigcup_{i=1}^k P_i \not\supseteq D$?

Our motivation for studying the above covering problems comes from two other related problems on polytopes. The first is the well-known *vertex-enumeration* problem of finding the vertices of a polytope given its facet defining inequalities, to be described in more details in the next section. The second problem is to check whether the union of a given set of polytopes is convex. Bemporad, Fukuda and Torrisi [3] gave polynomial-time algorithms for checking if the union of $k = 2$ polyhedra is convex, and if so finding this union, no matter whether they are given in $\mathcal{V}$ or $\mathcal{H}$ representations. They also gave necessary and sufficient conditions for the union of a finite number of convex polytopes in $\mathbb{R}^d$ to be convex, and asked whether these conditions can be used to design a polynomial time algorithm for checking if the union is convex. Bárány and Fukuda give slightly stronger conditions in [2]. It will follow from our results that, if both d and k are part of the input, then these conditions can not be checked in polynomial time unless P=NP.

Unless otherwise specified, all the cones considered throughout the paper will be assumed to be pointed, i.e., contain no lines, or equivalently, have a well defined apex, namely the origin. As we shall see, the complexity of the above problem depends on how the cones are represented, and whether they are disjoint or not. We consider 3 different factors, namely:

(f1) whether the cones in $\mathcal{C}$ are given in $\mathcal{V}$- or $\mathcal{H}$-representations, or both representations ($\mathcal{VH}$),

(f2) what the set D is: we consider $D = \mathbb{R}^d$ and $D = \mathbb{R}^t$ for some arbitrary $k \leq d$.

(f3) whether the cones in $\mathcal{C}$ are

- (f3)-(I): pairwise disjoint in the interior and intersect only at faces;
- (f3)-(II): pairwise disjoint in the interior , but can intersect anywhere on the boundaries; and
- (f3)-(III): not necessarily pairwise disjoint.

We denote by ConeCover$[F1, F2, F3]$ the different variants of the problem, where $F1 \in \{\mathcal{V}, \mathcal{H}\}$, $F2 \in \{\mathbb{R}^t, \mathbb{R}^d\}$ and $F3 \in \{I, II, III\}$ describes cases (f1)-(I), (f2)-(II), and (f3)-(III). Our results are summarized in Table 1.

*Max Planck Institut für Informatik, Saarbrücken, D-66123 Germany
†Universität des Saarlandes, Saarbrücken, D-66123 Germany
[1]possibly after moving first the polytope so that its relative interior contains the origin

	$\mathbb{R}^d$			$\mathbb{R}^t$		
	I	II	III	I	II	III
$\mathcal{V}$	VE-hard	VE-hard	NPC	NPC	NPC	NPC
$\mathcal{H}$	P	?	NPC	P	?	NPC
$\mathcal{VH}$	P	P	NPC	P	?	NPC

Table 1: Complexity of Cone Covering problem for various input representations.

2 Results

Converting the $\mathcal{H}$-representation of a polytope to its $\mathcal{V}$-representation and vice versa, is a well studied problem. Despite years of research, it is neither known if an output-sensitive algorithm exists for this problem, nor is it known to be NP-hard. The following decision version of this problem is known to be equivalent to the enumeration problem [1].

VertEnum(P, V): Given an $\mathcal{H}$-polytope $P \subseteq \mathbb{R}^d$ and a subset of its vertices $V \subseteq \mathcal{V}(P)$, check whether $P = \text{conv}(V)$.

Let P be the polytope defined as $\{x | Ax \leq \mathbf{1}\}$, where $A \in \mathbb{R}^{m\times d}$. Every rational polytope can be brought into this form by moving the origin in its relative interior and scaling the normals of the facet-defining hyperplanes appropriately. For any vertex v of P, consider the cone of all vectors c such that v is the solution of the following linear program: $\max c^T x$ s.t. $Ax \leq 1$. For every vertex v of P, this cone is uniquely defined. We call this cone the *maximizer cone* of v. Such a maximizer cone can be defined for every proper face of a polytope. The union of all such cones is also known as the *normal fan* of a polytope [8]. It is easy to see that if A' is the maximal subset of rows of A such that $A'v = 1$, then the maximizer cone of v is the conic hull of the rows of A' treated as vectors in $\mathbb{R}^d$.

Theorem 1 *Problem* ConeCover$[\mathcal{V}, \mathbb{R}^d, I]$ *is* VertEnum*-hard.*

Proof. Given an $\mathcal{H}$-polytope P and a subset of its vertices V, the $\mathcal{V}$-representation of the maximizer cone for each vertex in V can be computed easily from the facets of P. Clearly, the union of these cones covers $\mathbb{R}^d$ if and only if $P = conv(V)$. To see this, note that if $P \neq conv(V)$ then P has a vertex v not in V and any vector in the relative interior of the maximizer cone of v does not lie in any of the cones corresponding to the given vertices. □

Theorem 2 *Problem* ConeCover$[\mathcal{V}, \mathbb{R}^t, I]$ *is NP-complete.*

Proof. ConeCover$[\mathcal{V}, \mathbb{R}^t, \text{I}]$ is clearly in NP. Now, given an $\mathcal{H}$-polytope $P \subset \mathbb{R}^d$, an affine subspace $\mathbb{R}^t$ and a $\mathcal{V}$-polytope $Q \subset \mathbb{R}^k$, it is NP-complete to decide whether Q is the projection of P onto the given subspace [7]. We give a polynomial reduction from this problem to ConeCover$[\mathcal{V}, \mathbb{R}^t, \text{I}]$.

Every vertex v of Q is an image of some (possibly more than one) vertices of P. If this is not the case then Q clearly can not be the projection of P. Since the vertices of Q are known this condition can be checked in polynomial time. To see why this is true, consider a vertex v of Q and choose any direction α in the affine space of Q such that $\alpha^T x$ is maximized at v for all points in Q. If we use the same vector α as objective function over the points in P then the maximum is achieved at the face containing all vertices whose image under projection is v.

Now, Pick any such vertex and call it v'. We associate the maximizer cone of v' with v and refer to it as $\mathcal{C}(v)$. The $\mathcal{V}$-representation of $\mathcal{C}(v)$ for every vertex v of Q can be easily computed from the matrix A of the normals of facet defining hyperplanes of P.

It is not difficult to see that if Q is not the projection of P onto the given subspace $\mathbb{R}^t$, then one can find a direction c parallel to the given subspace such that a vertex that maximizes $c^T x$ in P is such that its projection is a vertex of the projection of P but not of Q. Hence, the union of cones $\mathcal{C}(v)$ for each vertex v of Q covers $\mathbb{R}^t$ if and only if Q is the projection of P. Also, all these cones intersect each other only at some proper face. □

For a given set of $\mathcal{H}$-cones, if the union does not cover $\mathbb{R}^d$ then there is a facet with normal $a \in \mathbb{R}^d$, of at least one of these cones such that picking a point p in the interior of this facet, $p + \epsilon a$ lies outside every cone, for some $\epsilon > 0$. Let us call this facet a *witness facet*, and p a *witness point* of the fact that $\mathbb{R}^d$ is not covered.

Theorem 3 *There is a polynomial time algorithm for solving* ConeCover$[\mathcal{H}, \mathbb{R}^d, I]$.

Proof. If the cones are allowed to intersect only at common faces, then every point in the interior of a witness facet is a witness point. Thus, one can determine in polynomial time whether the union of the given cones cover $\mathbb{R}^d$ or not, by picking a point in the interior of every facet, with normal a, of every cone and using linear programming to check if $p + \epsilon a$ lies outside every cone for sufficiently small $\epsilon > 0$. □

Theorem 4 *There is a polynomial time algorithm for* ConeCover$[\mathcal{VH}, \mathbb{R}^d, II]$.

Proof. It is easy to see that if the cones are allowed to intersect only on the boundary, and if the union of the given cones does not cover $\mathbb{R}^d$, then the extreme rays of any (possibly non-convex) "hole" are also the extreme rays of some cone. For any such extreme ray w, if one considers a d-dimensional ball of radius ϵ centered at some point on w, then for small enough ϵ some part of this ball is not covered by any of the given cones.

Consider all the halfspaces $\{x|\ ax \leq 0\}$ corresponding to the facets of the input cones that contain w, *i.e.*, $aw = 0$. Let A be the matrix with each row the normal vector of such a halfspace. The union of the given cones does not cover $\mathbb{R}^d$ if and only if $\{x|\ Ax \geq 0\}$ defines a full-dimensional region. This can be easily checked via linear programming. ☐

Fact 1 *For any $t \in \mathbb{N}$, we can write $\mathbb{R}^t = \cup_{i=1}^{k+1} R_i$, where $R_1, \ldots, R_{k+1}$ are pointed cones, pairwise-disjoint in the interior, whose $\mathcal{H}$- and $\mathcal{V}$-representations can be found in in polynomial time.*

Let $C_1 = \{x \in \mathbb{R}^m \mid A_1 x \leq \mathbf{0}\} = \text{cone}(S_1)$ and $C_2 = \{x \in \mathbb{R}^n \mid A_2 x \leq \mathbf{0}\} = \text{cone}(S_2)$, where $A_1 \in \mathbb{R}^{l \times m}, A_2 \in \mathbb{R}^{r \times n}$ and $S_1 \subseteq \mathbb{R}^m, S_2 \subseteq \mathbb{R}^n$, be two polyhedral cones. The *direct-sum* of C_1 and C_2, is defined as:

$$\begin{aligned} C_1 \oplus C_2 &= \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n |\ A_1 x \leq \mathbf{0},\ A_2 y \leq \mathbf{0}\} \\ &= \text{cone}\left(\left\{\begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} : v \in S_1\right\} \bigcup \left\{\begin{pmatrix} \mathbf{0} \\ v \end{pmatrix} : v \in S_2\right\}\right) \end{aligned}$$

Theorem 5 *Problem* ConeCover$[\mathcal{VH}, \mathbb{R}^d, III]$ *is NP-complete.*

Proof. Clearly the problem is in NP since a direction exists outside the union of the given cones if they do not cover $\mathbb{R}^d$. We can easily check if such a given direction indeed lies outside each of the cones since the facets of each cone are known. For proving its NP-hardness, we use a reduction from the following problem:

Sat$(V, \mathcal{F}, \mathcal{G})$: Given a finite set V and two hypergraphs $\mathcal{F}, \mathcal{G} \subseteq 2^V$, is there a set $X \subseteq V$ such that:

$$X \not\supseteq F \text{ for all } F \in \mathcal{F} \text{ and } X \not\subseteq G \text{ for all } G \in \mathcal{G}. \quad (1)$$

When $\mathcal{F} = \mathcal{G}$, this problem is called the *saturation problem* in [4], where it is proved to be NP-complete. Given $\mathcal{F}, \mathcal{G} \subseteq 2^V$, we construct two families of cones $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ in $\mathbb{R}^V$, such that there is a point $x \in \mathbb{R}^V \setminus (\mathcal{C}_\mathcal{F} \cup \mathcal{C}_\mathcal{G})$ if and only if $\mathcal{F}$ and $\mathcal{G}$ are not saturated (i.e. there is a set $X \subseteq V$ satisfying (1)).

For $X \subseteq V$, denote respectively by $\mathbb{R}^X_\geq$ and $\mathbb{R}^X_\leq$ the cones $\text{cone}\{\mathbf{e}_i : i \in X\} = \{x \in \mathbb{R}^X : x \geq \mathbf{0}\}$ and $\text{cone}\{-\mathbf{e}_i : i \in X\} = \{x \in \mathbb{R}^X : x \leq \mathbf{0}\}$, where $\mathbf{e}_i$ denotes the standard ith unit vector. Let $\overline{X} = V \setminus X$, and $\bigcup_{i=1}^{|X|+1} R_i(X) = \mathbb{R}^X$ be the partition of $\mathbb{R}^X$ given by Fact 1. For each $F \in \mathcal{F}$, we define $|V| - |F| + 1$ cones $C^i_F = \mathbb{R}^F_\geq \oplus R_i(\overline{F})$, for $i \in [|\overline{F}| + 1]$, and for each $G \in \mathcal{G}$, we define $|G|+1$ cones $C^i_G = \mathbb{R}^{\overline{G}}_\leq \oplus R_i(G)$, for $i \in [|G|+1]$.

Finally, we let $\mathcal{C}_\mathcal{F} = \{C^i_F : F \in \mathcal{F},\ i \in [|\overline{F}| + 1]\}$, $\mathcal{C}_\mathcal{G} = \{C^i_G : G \in \mathcal{G},\ i \in [|G| + 1]\}$, and $\mathcal{C} = \mathcal{C}_\mathcal{F} \cup \mathcal{C}_\mathcal{G}$. Then it is not difficult to see that all the cones in $\mathcal{C}$ are pointed.

Suppose that $X \subseteq V$ satisfies (1). Define $x \in \mathbb{R}^V$ by

$$x_i = \begin{cases} 1, & \text{if } i \in X, \\ -1, & \text{if } i \in V \setminus X. \end{cases}$$

Then $x \notin \cup_{C \in \mathcal{C}} C$. Indeed, if $x \in C^i_F$, for some $F \in \mathcal{F}$ and $i \in [|\overline{F}| + 1]$, then $x_i \geq 0$ and hence $x_i = 1$, for all $i \in F$, implying that $X \supseteq F$. Similarly, if $x \in C^i_G$, for some $G \in \mathcal{G}$ and $i \in [|G| + 1]$, then $x_i \leq 0$ and hence $x_i = -1$, for all $i \in \overline{G}$, implying that $X \subseteq G$.

Conversely, suppose that $x \in \mathbb{R}^V \setminus \mathcal{C}$. Let $X = \{i \in V :\ x_i \geq 0\}$. Then we claim that X satisfies (1). Indeed, if $X \supseteq F$ for some $F \in \mathcal{F}$, then $x_i \geq 0$ for all $i \in F$, and hence there exists an $i \in [|\overline{F}| + 1]$ such that $x \in C^i_F$ (since the cones $R_1(\overline{F}), \ldots, R_{|\overline{F}|+1}(\overline{F})$ cover $\mathbb{R}^{\overline{F}}$). Similarly, if $X \subseteq G$ for some $G \in \mathcal{G}$, then $x_i < 0$ for all $i \in \overline{G}$, and hence there exists an $i \in [|G|+1]$ such that $x \in C^i_G$. In both cases we get a contradiction. ☐

Corollary 1 ConeCover$[\mathcal{V}, \mathbb{R}^d, III]$, ConeCover$[\mathcal{H}, \mathbb{R}^d, III]$ *and* ConeCover$[\mathcal{H}, \mathbb{R}^t, III]$ *are all NP-complete.*

Proof. NP-completeness of ConeCover$[\mathcal{V}, \mathbb{R}^d, \text{III}]$ and ConeCover$[\mathcal{H}, \mathbb{R}^d, \text{III}]$ follow immediately from Theorem 5. NP-completeness of ConeCover$[\mathcal{H}, \mathbb{R}^t, \text{III}]$ is an immediate consequence of the NP-hardness of ConeCover$[\mathcal{H}, \mathbb{R}^d, \text{III}]$ and the fact that for an $\mathcal{H}$-cone, the intersection of this cone with any affine subspace can be computed easily. ☐

An interesting special case of problem Sat is when the hypergraphs $\mathcal{F}$ and $\mathcal{G}$ are *transversal* to each other:

$$F \not\subseteq G \text{ for all } F \in \mathcal{F} \text{ and } G \in \mathcal{G}, \quad (2)$$

in which case, the problem is known as the *hypergraph transversal problem*, denoted HyperTrans. Even though the complexity of this problem is still open, it is unlikely to be NP-hard since there exist algorithms [5] that solve the problem in quasi-polynomial time $m^{o(\log m)}$, where $m = |\mathcal{F}| + |\mathcal{G}| + |V|$. Improving this to a polynomial bound is a standing open question. We observe from our reduction in Theorem 5 that ConeCover includes HyperTrans as a special case.

Corollary 2 *Consider a family of cones $\mathcal{C}$ that can be partitioned into two families $\mathcal{C}_1$ and $\mathcal{C}_2$ such that*

$$\text{int}(C_1) \cap \text{int}(C_2) = \emptyset, \quad \textit{for all } C_1 \in \mathcal{C}_1 \textit{ and } C_2 \in \mathcal{C}_2. \quad (3)$$

Then ConeCover$(\mathcal{C}, \mathbb{R}^d)$ *is* HyperTrans-*hard.*

Proof. We note in the construction used on the proof of Theorem 5 that if the hypergraphs $\mathcal{F}$ and $\mathcal{G}$ satisfy (2), then the families of cones $\mathcal{C}_\mathcal{F}$ and $\mathcal{C}_\mathcal{G}$ satisfy (3). Indeed, if $x \in C_F^i \cap C_G^j$, for some $F \in \mathcal{F}$, $i \in [|\overline{F}|+1]$, $G \in \mathcal{G}$, and $j \in [|G|+1]$, then $x_k \geq 0$ for all $k \in F$ and $x_k \leq 0$ for all $k \in \overline{G}$. Thus for any $k \in F \setminus G$ (which must exist by (2)), we have $x_k = 0$, implying that x is not an interior point in either C_F^i or C_G^j. □

Freund and Orlin [6] proved that, for an $\mathcal{H}$-polytope P and a $\mathcal{V}$-polytope Q, checking if $Q \supseteq P$ is NP-hard. For all other representations of P and Q, checking $P \subseteq Q$ can be done by solving a linear program. Here we can show that the union version of this problem is hard, no matter how the polytopes are represented.

Corollary 3 *Given a set of $\mathcal{H}$-polytopes $\mathcal{P} = \{P_1, \ldots, P_k\}$ and an $\mathcal{H}$-polytope Q, problem* PolytopeCover$(\mathcal{P}, Q)$ *is NP-hard.*

Proof. We give a reduction from problem ConeCover$[\mathcal{H}, \mathbb{R}^d, \text{III}]$ which is NP-hard by Theorem 5. Let S_d be a "shifted" simplex in $\mathbb{R}^d$ such that $\mathbf{0} \in \text{int}(S_d)$. Given cones $C_1, \ldots, C_k$, we define polytopes $P_1, \ldots P_k$, where $P_i = C_i \cap S_d$. Given the $\mathcal{H}$-representations of C_i, we can compute the $\mathcal{H}$-representations of P_i in polynomial time using linear programming (LP) for removing possible redundancies.

Now one can easily see that $\cup_{i=1}^k C_i = \mathbb{R}^d$ iff $\cup_{i=1}^k P_i = S_d$. □

Corollary 4 *Given a set of $\mathcal{V}$-polytopes $\mathcal{P} = \{P_1, \ldots, P_k\}$ and a $\mathcal{V}$-polytope Q, problem* PolytopeCover$(\mathcal{P}, Q)$ *is NP-hard.*

Proof. We give a reduction from problem ConeCover$[\mathcal{V}, \mathbb{R}^d, \text{III}]$ which is NP-hard by Theorem 5. Recall that in the proof of Theorem 5, for each hyperedge F we construct a set of pointed cones $C_F^i = \mathbb{R}^F_\geq \oplus R_i(\overline{F})$, for $i \in [|\overline{F}|+1]$. Instead of constructing multiple cones for each hyperedge let us just consider one cone $C_F = \mathbb{R}^F_\geq \oplus \mathbb{R}^{|\overline{F}|}$ per hyperedge. Similarly for the cones corresponding to the hypergraph $\mathcal{G}$. It is clear that $C_F = \cup_{i=1}^{|\overline{F}|+1} C_F^i$. Note that each such cone is not pointed but instead has a pointed part $\mathbb{R}^F_\geq$ corresponding to the vertices in the hyperedge F and the affine space $\mathbb{R}^{|\overline{F}|}$ corresponding to the vertices not in F. Also, $\mathbb{R}^F_\geq$ is one orthant in $\mathbb{R}^{|F|}$.

For such cones checking whether the union covers $\mathbb{R}^d$ or not is NP-hard as well (see proof of Theorem 5). Now consider the d-dimensional cross-polytope β_d, and let $C_1, \ldots, C_k$ be the cones constructed above. The cross polytope β_d contains the origin in its interior, and the vertices of $P_i = \beta_d \cap C_i$ for each cone constructed above can be easily computed. It is also easy to see that $\cup_{i=1}^k C_i = \mathbb{R}^d$ iff $\cup_{i=1}^k P_i = \beta_d$. □

Theorem 6 *Given a set of rational convex polytopes $P_1, \ldots, P_k \subseteq \mathbb{Q}^d$, it is coNP-complete to check if their union is convex, for both $\mathcal{H}$ and $\mathcal{V}$-representations of the input polytopes.*

Proof. First we show that the problem is in coNP. Let $Q = \cup_{i=1}^k P_i$. Given two points $x, y \in Q$, we want to verify that the line segment $[x, y] \stackrel{\text{def}}{=} \{\lambda x + (1-\lambda) y | \lambda \in [0,1]\} \not\subseteq Q$. This can be done by iterating the algorithm for two polytopes in [3]: 1. let P be the polytope P_i such that $x \in P_i$; 2. find the (last) point $z \in P$ on the ray $\{x + \lambda(y - x) | \lambda \geq 0\}$ such that λ is maximized; 3. if there is another polytope P_j such that $z \in P_j$, then set $P \leftarrow P_j$, $x \leftarrow z$, and go to step 2 else output "No" and halt; 4. if $x = y$ then output "Yes" and halt. The reader can verify that all the above steps can be implemented in polynomial time, given an oracle for LP, and no matter how the polytopes are represented.

Consider the $\mathcal{H}$-representation first. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ and S_d be the polytopes used in the construction in Corollary 3. We now reduce problem PolytopeCover$(\mathcal{P}, S_d)$ to checking if the union of a given set of polytopes is convex. Using an algorithm for the latter problem, we can check if $P = \cup_{i=1}^k P_i$ is convex. If the answer is "No", we conclude that $P \neq S_d$. Otherwise, since $P \subseteq S_d$, either $P = S_d$, or there is hyperplane separating a vertex of S_d from P. The latter condition can be checked in polynomial time by solving k linear programs for each vertex.

For the $\mathcal{V}$-representation the same argument as above works if we use β_d instead of S_d. □

References

[1] D. Avis, D. Bremner, and R. Seidel. How good are convex hull algorithms? *Comput. Geom.*, 7:265–301, 1997.

[2] I. Bárány and K. Fukuda. A case when the union of polytopes is convex. *Linear Algebra and its Applications*, 397(6):381–388, 2005.

[3] A. Bemporad, K. Fukuda, and F. D. Torrisi. Convexity recognition of the union of polyhedra. *Comput. Geom.*, 18(3):141–154, 2001.

[4] T. Eiter and G. Gottlob. Identifying the minimal transversals of a hypergraph and related problems. *SIAM J. Comput.*, 24(6):1278–1304, 1995.

[5] M. L. Fredman and L. Khachiyan. On the complexity of dualization of monotone disjunctive normal forms. 21:618–628, 1996.

[6] R. M. Freund and J. B. Orlin. On the complexity of four polyhedral set containment problems. *Mathematical Programming*, 33(2):139–145, 1985.

[7] H. R. Tiwary. On computing the shadows and slices of polytopes. *CoRR*, abs/0804.4150, 2008.

[8] G. M. Ziegler. *Lectures on Polytopes*. Graduate Texts in Mathematics, No. 152. Springer-Verlag, Berlin, 1995.