On Directed Graphs with an Upward Straight-line Embedding into Every Point Set

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Abstract

In this paper we study the problem of computing an *upward straight-line embedding* of a directed graph G into a point set S, i.e. a planar drawing of G such that each vertex is mapped to a point of S, each edge is drawn as a straight-line segment, and all the edges are oriented according to a common direction. We characterize the family of directed graphs that admit an upward straight-line embedding into every one-side convex point set, that is, into every point-set such that the top-most and the bottom-most points are adjacent in the convex hull of the point set. Also we show how to construct upward straight-line embeddings for a sub-class of directed paths when the point set is in general position.

1 Introduction

Given a planar graph G and a point set S in the plane, a *straight-line embedding* of G into S is a mapping of each vertex of G to a point of S and of each edge of G to a straight-line segment between its endpoints such that no two edges intersect. Several variants of this problem have been studied in the Graph Drawing and Computational Geometry fields, from both a combinatorial and an algorithmic point of view. Gritzmann *et al.* [6] proved that the class $\mathcal{F}$ of undirected graphs that admit a straight-line embedding into *every* point set in general position coincides with the class of *outerplanar graphs*. An algorithm to compute a straight-line embedding of an outerplanar graph in $O(n \log^3 n)$-time has been presented by Bose [2], while an optimal $\Theta(n \log n)$-time algorithm [3] is known for trees. The problem of deciding whether a planar graph admits a straight-line embedding into a given point set has been proved to be $\mathcal{NP}$-hard [4].

The version of the problem in which G is an acyclic directed graph has received less attention in the literature. Drawings of directed acyclic graphs are usually required to be *upward*, i.e., all edges flow in a common predefined direction according to their orientation. We will assume, w.l.o.g., that such a direction is the one of increasing y-coordinates. Preliminary results in this scenario have been proved by Estrella-Balderrama *et al.* [5]. They prove that no biconnected directed graph admits an upward straight-line embedding into every point set in convex position; they provide a characterization of the Hamiltonian directed graphs that admit upward straight-line embeddings into every point set in general or in convex position. Finally, they describe how to construct upward straight-line embeddings of directed paths into convex point sets and prove that for directed trees such embeddings do not always exist. However, the directed counterpart of the result by Gritzmann *et al.* [6], i.e., a characterization of the family $\overrightarrow{\mathcal{F}}$ of directed acyclic graphs that admit an upward straight-line embedding into *every* point set in general position is still missing. In this paper we study such a problem and prove the following results.

- We characterize the family of directed graphs that admit an upward straight-line embedding into every one-side convex point set, i.e., a convex point set S in which the top-most and the bottom-most points are adjacent in the convex hull of S (Section 3).

- We show how to construct upward straight-line embeddings of *regular paths* into point sets in general position (Section 4). Regular paths are a family of directed paths such that, considering the vertices in the order they appear on the path, every sink is followed by a source or vice versa.

For reasons of space some proofs are sketched or omitted. A complete version of this paper can be found in [1].

2 Preliminaries

A graph G is *outerplanar* if it admits a planar embedding in which all vertices are incident to the outer face. Such an embedding is called *outerplanar embedding*. The maximal biconnected subgraphs of a graph G are its *blocks*. The *block-cutvertex tree*, or BC-tree, of a connected graph G is a tree with a B-node for each block of G and a C-node for each cutvertex of G. Edges in the BC-tree connect each B-node μ to the C-nodes associated with the cutvertices in the block of μ. In the

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following we identify a block (resp. a cutvertex) with the B-node (resp. the C-node) associated with it.

Let G be a directed graph; a vertex v of G is a *source (sink)* if v has no incoming (outgoing) edges. An *upward planar directed graph* is a directed graph that admits a planar drawing such that each edge is represented by a curve monotonically increasing in the y-direction. A *Hamiltonian* directed graph G is a directed graph containing a path $(v_1, \dots, v_n)$ passing through all vertices of G such that edge (v_i, v_{i+1}) is directed from v_i to v_{i+1}, for each $1 \leq i \leq n-1$.

A point set in the plane is in *general position* if no three points lie on the same line. The *convex hull* $CH(S)$ of a point set S is the point set that can be obtained as a convex combination of the points of S. A point set is in *convex position* if no point is in the convex hull of the others. As in [5], we will assume that no two points of any point set have the same y-coordinate. Such an assumption avoids the *a priori* impossibility of drawing an edge between two specified points of the point set. Then, the points of any point set S can be totally ordered by increasing y-coordinate. We refer to the i-th point as the one which has $i-1$ points with smaller y-coordinates. Let $p_m(S)$ and $p_M(S)$ be the first and the last point of S, respectively. In a convex point set S two points are *adjacent* if the segment between them is on the border of $CH(S)$. We say that S is a *one-side* convex point set if $p_M(S)$ and $p_m(S)$ are adjacent.

3 Upward Straight-line Embeddings of Graphs

In this section we characterize the family $\overrightarrow{\mathcal{F}}_1$ of directed graphs that admit an upward straight-line embedding into every one-side convex point set. Notice that since one-side convex point sets are a special case of point sets in general position, then $\overrightarrow{\mathcal{F}} \subseteq \overrightarrow{\mathcal{F}}_1$.

First, we give some properties that must be satisfied by each block B of a directed graph G in order to admit an upward straight-line embedding into every one-side convex point set.

Pr1: B is an outerplanar graph.

Pr2: B has an outerplanar embedding such that the boundary C of the external face consists of a Hamiltonian directed path $(s = v_1, \dots, v_k = t)$ and of the edge (s,t).

Pr3: Edges not belonging to C are such that, if an edge (v_{i_1}, v_{j_1}) belongs to B, then no edge (v_{i_2}, v_{j_2}) belongs to B, with $i_1 < i_2 < j_1 < j_2$.

The necessity of **Pr1** and of **Pr2** can be easily proved (see also [5]). We prove the necessity of **Pr3**. Observe that, in any upward straight-line embedding of B into a one-side convex point set, the order of the vertices of B by increasing y-coordinates is the same as their order in the Hamiltonian directed path $(s = v_1, \dots, v_k = t)$, otherwise an edge (v_i, v_{i+1}) would not be upward. Then, if two edges (v_{i_1}, v_{j_1}) and (v_{i_2}, v_{j_2}) belong to B, with $i_1 < i_2 < j_1 < j_2$, they cross.

We call *regular* every block satisfying Properties **Pr1**, **Pr2**, and **Pr3**. By **Pr2** each regular block B has exactly one source and one sink, hence in the following, we will talk about *the source* of B and *the sink* of B.

Let $\mathcal{T}$ be the BC-tree of a connected directed graph G. Consider a B-node B of $\mathcal{T}$ and a C-node c adjacent to B. We say that c is *extremal* for B if it is either the source or the sink of B, c is *non-extremal* for B otherwise. In the following, we build an auxiliary directed tree $\mathcal{T}'$ starting from $\mathcal{T}$ (see Fig. 1). A node μ of $\mathcal{T}'$ corresponds to a connected subtree $\mathcal{S}$ of $\mathcal{T}$ which is maximal with respect to the following property: A cutvertex $c_{1,2}$ that is adjacent in $\mathcal{S}$ to two B-nodes B_1 and B_2 is extremal for both B_1 and B_2. An edge of $\mathcal{T}'$ directed from μ to ν corresponds to a cutvertex which is non-extremal for a block associated with μ and extremal for a block associated with ν.

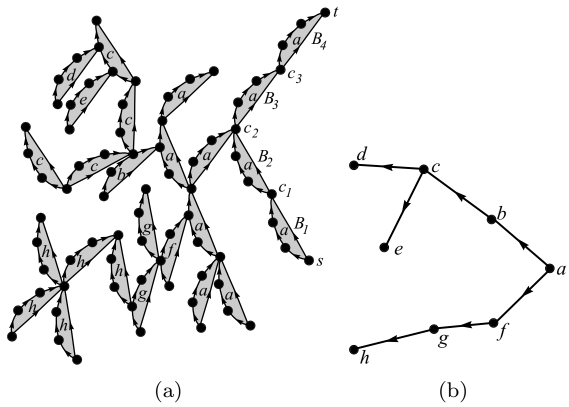

Figure 1: (a) A directed graph G. (b) Auxiliary tree $\mathcal{T}'$ built from the BC-tree $\mathcal{T}$ of G.

Now we prove the main result of this section.

Theorem 1 *An n-vertex connected directed graph G admits an upward straight-line embedding into every one-side convex point set of size n if and only if the following conditions are satisfied: (1) Each block of G is regular; (2) No cutvertex shared by two blocks is non-extremal for both of them; (3) Every node of $\mathcal{T}'$ has at most one incoming edge.*

Proof sketch: The necessity of Condition (1) has been already proved before the definition of regular block. To prove the necessity of Condition (2), we show that, given any one-side convex point set S, if G contains a cutvertex $c_{1,2}$ that is non-extremal for two blocks B_1 and B_2 then G has no upward straight-line embedding into S. Denote by P_1 and P_2 the Hamiltonian directed paths of B_1 and B_2, respectively. Denote by s_1 and t_1 (s_2 and t_2) the source and the sink of B_1 (B_2), respectively. Further, denote by $P(s_1, c_{1,2})$ ($P(s_2, c_{1,2})$) the subpath of P_1 (P_2) between s_1 and $c_{1,2}$ (s_2 and $c_{1,2}$). Finally, denote by v_1 (v_2) the vertex of B_1 coming immediately before $c_{1,2}$ in P_1 (P_2). Suppose, for a contradiction, that an upward straight-line embedding of G into S exists. Also suppose, w.l.o.g., that s_1 is mapped to a point of S with y-coordinate smaller than the one of the point of S where s_2 is mapped to. Then, $P(s_1, c_{1,2})$ and $P(s_2, c_{1,2})$ do not cross only if $P(s_2, c_{1,2})$ is embedded entirely into points of S between the points where v_1 and $c_{1,2}$ are mapped to. Since the embedding is upward, vertex t_2 is mapped to a point of S with y-coordinate greater than the one of $c_{1,2}$. It follows that edge (s_2, t_2) crosses edge $(v_1, c_{1,2})$. We prove the necessity of Condition (3). Suppose that $\mathcal{T}'$ contains edges (n_1, n^*) and (n_2, n^*) and suppose, for a contradiction, that an upward straight-line embedding of G into a one-side convex point set S exists. By definition of $\mathcal{T}'$, there exist two blocks B_1 and B_2 of G associated with nodes n_1 and n_2 of $\mathcal{T}'$, respectively, and there exist two blocks B_3 and B_4 associated with node n^* of $\mathcal{T}'$ such that a cutvertex $c_{1,3}$ of G is non-extremal for B_1 and extremal for B_3, and a cutvertex $c_{2,4}$ of G is non-extremal for B_2 and extremal for B_4. Note that it is possible that $B_3 = B_4$, while $B_1 \neq B_2$ and $c_{1,3} \neq c_{2,4}$. Indeed, if $B_1 = B_2$, then n_1 and n_2 would not be distinct nodes of $\mathcal{T}'$; further, if $c_{1,3} = c_{2,4}$, then $c_{1,3}$ would be non-extremal for both B_1 and B_2, violating Condition 2 and hence contradicting the fact that an upward straight-line embedding of G into S exists. Denote by P_1 and P_2 the Hamiltonian directed paths of B_1 and B_2, respectively. Denote by s_1 and t_1 (s_2 and t_2) the source and the sink of B_1 (B_2), respectively. Consider any upward straight-line embedding of B_1 and B_2 into S. In order for the embedding to be planar, two are the cases, up to a renaming of B_1 and B_2: Either the vertices of B_2 are mapped to points of S whose y-coordinates are all greater than the ones of the points of S where the vertices of B_1 are mapped to, or the vertices of B_2 are mapped to points of S whose y-coordinates are all between the ones of two consecutive vertices of P_1, say v_i and v_{i+1}. In both cases $c_{1,3}$ has y-coordinate either greater than the y-coordinates of all the vertices of B_2 or smaller than the y-coordinates of all the vertices of B_2. This implies that $c_{1,3}, s_2, c_{2,4}$, and t_2 are ordered according to their y-coordinates either in this order or in the order $s_2, c_{2,4}, t_2, c_{1,3}$. The set of blocks associated with n^* contains vertices $c_{1,3}$ and $c_{2,4}$ and thus it contains a path P^* between $c_{1,3}$ and $c_{2,4}$. Path P^* is composed by vertices all distinct from the vertices of B_1 (B_2), except for $c_{1,3}$ ($c_{2,4}$), otherwise B_1 and B_3 (B_2 and B_4) would not be distinct blocks of G. Hence, P^* intersects edge (s_2, t_2).

To prove the sufficiency of Conditions (1)–(3) we describe how to compute an upward straight-line embedding of G into a given one-side convex point set S. First, we decide a planar embedding $\mathcal{E}$ of G, that is, the order of the edges incident to each vertex and the outer face in the final embedding of G into S. Each block B_i of G with source s_i and sink t_i is embedded in such a way that the embedding is outerplanar and the external face is on the right-hand side when walking along edge (s_i, t_i) from s_i to t_i. We call such an embedding of B_i a *regular embedding* of B_i. Note that since the embedding of B_i is outerplanar, then the outer face of B_i is delimited by the Hamiltonian directed path of B_i and by edge (s_i, t_i). A *bimodal* embedding of a directed graph G is such that for each vertex of G the circular list of its incident edges can be partitioned into two (possibly empty) lists, one consisting of incoming edges and the other consisting of outgoing edges. The embedding $\mathcal{E}$ of G is set as follows: Consider the subgraph $\mathcal{T}_a$ of $\mathcal{T}$ whose blocks and cutvertices correspond to any node a of $\mathcal{T}'$ without incoming edges; choose a path $(B_1, c_1, B_2, c_2, \ldots, B_h)$ in $\mathcal{T}_a$ such that the source s of B_1 is a source of G, the sink c_i of B_i is the source of B_{i+1}, for each $1 \leq i \leq h-1$, and the sink t of B_h is a sink of G. It is possible to prove that such a path always exists. Then, the embedding $\mathcal{E}$ of G is any bimodal outerplanar embedding in which the embedding of each block is regular and path $(s, c_1, c_2, \ldots, c_{h-1}, t)$ has edges consecutive along the boundary of the outer face of $\mathcal{E}$. The embedding in Fig. 1 (a) satisfies the properties just described. In order to compute an upward straight-line embedding of G into S, we map the vertices of G to the points of S one at a time. The i-th mapped vertex of G is mapped to the i-th point of S. Let s be the source of G defined when deciding the planar embedding $\mathcal{E}$ of G. Starting from s we walk in clockwise direction on the boundary of the outer face of $\mathcal{E}$. A vertex v of G is mapped to a point of S when the algorithm visits v and for each edge (u, v), oriented from u to v, vertex u has already been mapped to a point of S. The computed drawing is straight-line and upward by construction; the proof of its planarity is omitted and can be found in [1]. $\square$

The described characterization can be easily exploited to design a polynomial-time algorithm testing whether a directed graph admits an upward straight-line embedding into every one-side convex point set or not.

It is also worth noting that a generalization of the characterization to non-connected graphs is easy, as it is possible to prove that an n-vertex directed graph admits an upward straight-line embedding into every one-side convex point set of size n if and only if each of its connected components admits an upward straight-line embedding into every one-side convex point set of size equal to the number of vertices of the component.

4 Upward Straight-line Embeddings of Paths

Since all directed paths admit an upward straight-line embedding into every convex point set [5], it is natural to ask whether the statement is also true for point sets in general position. The following theorem gives a partial answer to this question. Let $P = (v_1, \ldots, v_n)$ be a directed path. We say that P is *right-regular* if, for any sink $v_i \neq v_n$, v_{i+1} is a source of P. We say that P is *left-regular* if, for any sink $v_i \neq v_1$, v_{i-1} is a source of P. We say that P is *regular* if it is either right- or left-regular.

Theorem 2 *Every n-vertex regular path admits an upward straight-line embedding into every point set of size n in general position.*

Proof. Let P be a right-regular path $(v_1, \ldots, v_n)$. If P is left-regular, the proof is symmetric. Let $P_{h,k} \subseteq P$ be the subpath $(v_h, \ldots, v_k)$, where $1 \leq h \leq k \leq n$. Throughout the algorithm, we denote as p_i the point of S where v_i is mapped to and we denote as U_h the set $S \setminus \{p_j \mid j = 1, 2, \ldots, h\}$.

Let v_j be the sink of P such that j is minimum. If $j = n$, mapping v_i to the i-th point of S provides an upward straight-line embedding of P into S. Hence, assume that $j < n$. By definition, v_{j+1} is a source. Consider the subpath $P_{1,j+1}$. Each vertex v_i of $P_{1,j+1}$ for $i = 1, \ldots, j-1$ is mapped to the i-th point of S. Vertex v_{j+1} is mapped to point $p_{j+1} = p_m(U_{j-1})$, while p_j is mapped to the point of $CH(U_{j-1})$ that is adjacent to p_{j+1} and that is visible from p_{j-1} (if both the points of $CH(U_{j-1})$ adjacent to p_{j+1} are visible from p_{j-1}, then p_j is arbitrarily mapped to one of them). Note that, if $j = n-1$, the drawing is completed. If $j = 1$, then $P_{1,j+1} = (v_1, v_2)$; in this case v_2 is mapped to the point $p_2 = p_m(S)$ and vertex v_1 is mapped to one of the two points of $CH(S)$ that are adjacent to p_2. We recursively draw path $P_{j+1,n}$ into the point set U_j. Note that vertex v_{j+1} is considered twice, namely once when drawing $P_{1,j+1}$ and once when drawing $P_{j+1,n}$; however, when the drawing of $P_{j+1,n}$ is computed, v_{j+1} is placed on the bottom-most point of U_j, which is p_{j+1}. Therefore v_{j+1} is mapped twice to the same point.

The computed embedding is straight-line and upward by construction. We prove that it is planar. The proof is by induction on the number q of sinks. The case when $q = 1$ can be easily proved. Assume that $q > 1$ and let v_j, where $1 \leq j \leq n-1$, be the first sink encountered moving along P starting from v_1. The drawing of $P_{1,j-1}$ is trivially planar. We prove now that at least one of the two points of $CH(U_{j-1})$ adjacent to p_{j+1} is visible from p_{j-1}. Let p' and p'' be the two points of $CH(U_{j-1})$ adjacent to p_{j+1}. Let ℓ' be the line through p' and p_{j+1} and let ℓ'' be the line through p'' and p_{j+1}. Point p' is visible from all points below ℓ' and p'' is visible from all points below ℓ''. Since p_{j-1} is below p_{j+1} it is either below ℓ', or below ℓ'', or below both. This implies that at least one between p' and p'' is visible from p_{j-1} and therefore the algorithm always finds a point to map p_j. Edge (v_{j+1}, v_j) does not cross any other edge of $P_{1,j-1}$ because it is completely drawn above point p_{j-1}, and does not cross edge (v_{j-1}, v_j) because it shares an endvertex with such an edge; analogously, edge (v_{j-1}, v_j) does not cross any edge of $P_{1,j-2}$ because it is completely drawn above p_{j-2}, and does not cross edge (v_{j-2}, v_{j-1}) because it shares an endvertex with (v_{j-2}, v_{j-1}). Thus $P_{1,j+1}$ is planar. The drawing of $P_{j+1,n}$ is planar by induction and it is completely contained in $CH(U_j)$. The drawing of $P_{1,j-1}$ is completely contained in $CH(\{p_1, \ldots, p_{j-1}\})$. Convex hulls $CH(U_j)$ and $CH(\{p_1, \ldots, p_{j-1}\})$ are disjoint, since the points of U_j are all above p_{j-1}, hence the edges of $P_{1,j-1}$ do not cross with those of $P_{j+1,n}$. Further, edge (v_{j-1}, v_j) is external to both $CH(U_j)$ and $CH(\{p_1, \ldots, p_{j-1}\})$, and edge (v_j, v_{j+1}) is on the border of $CH(U_j)$ and external to $CH(\{p_1, \ldots, p_{j-1}\})$. □

5 Open Problems

The main open problems related to the topic of this paper remain the ones of characterizing the families of directed graphs that admit an upward straight-line embedding into every point set in general position and into every point set in convex position. We believe that determining whether every directed path admits an upward straight-line embedding into every point set in general position is a problem of its own interest.

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