Rigid Components of Random Graphs

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Abstract

We study the emergence of rigid components in an Erdős-Rényi random graph $\mathbb{G}(n, p)$, using the parameterization $p = c/n$ for a fixed constant $c > 0$. We show that for all $c > 0$, almost surely all rigid components have size 2, 3 or $\Omega(n)$; for $c > 4$, we show that almost surely there is a rigid component of size at least $n/10$.

1 Introduction

The problem of the phase transition between liquid and solid states of glasses is an important open problem in material physics [1]. Glasses are highly disordered solids that undergo a rapid transition as they cool.

To study the phase transition, Thorpe [8] proposed a *geometric* model for the glass problem, in which bonds between the atoms are viewed as **fixed-length bars** (the bonds) connected by **universal joints** (the atoms) with full rotational degrees of freedom. Such a structure is called a **planar bar-and-joint framework** (shortly bar-joint framework, or simply framework), and these are fundamental objects of study in the field of **combinatorial rigidity**.

A bar-joint framework is **rigid** if the only continuous motions of the joints preserving the lengths and connectivity of the bars are rigid motions of the plane, and otherwise it is **flexible**. When a framework is flexible, it decomposes uniquely into inclusion-wise maximal rigid substructures which are called **rigid components**; a rigid component is non-trivial if it is larger than a single edge. In the planar case, the celebrated Maxwell-Laman Theorem [10] gives a complete characterization of *generically* minimally rigid bar-joint frameworks in terms of a combinatorial condition, which allows rigidity properties to be studied in terms of efficiently checkable graph properties.

The sequence of papers [3, 7, 8, 17, 18] studies the emergence of *large* rigid subgraphs in graphs generated by various stochastic processes, with the edge probabilities and underlying topologies used to model the temperature and chemical composition of the system. Two important observations are that: (1) very large rigid substructures emerge very rapidly; (2) the transition appears to occur slightly below average degree 4 in the the planar bar-joint model.

Main result and novelty

In this paper, we study the emergence of rigid components in random graphs generated by a simple, well-known stochastic process: the Erdős-Rényi random graph model $\mathbb{G}(n, p)$, in which each edge is included with probability p, independently. We consider edge probabilities of the form $p = c/n$, where c is a fixed constant, and consider the size of the largest rigid components in $\mathbb{G}(n, p)$.

Theorem 1 (**Size and emergence of a large rigid component**). *Let $c > 0$ be a constant. Almost surely, all rigid components in $\mathbb{G}(n, c/n)$ span 2, 3, or $\Omega(n)$ vertices. If $c > 4$, then almost surely there is a rigid component of size at least $n/10$.*

A random graph has a property almost surely if the probability of $\mathbb{G}(n, p)$ having it tends to one as $n \to \infty$.

To the best of our knowledge, this is the first proven result on the emergence of rigid components in random graphs that have, almost surely, close to $2n - 3$ edges (the number required for minimal rigidity) but *no other special assumptions*, such as being d-regular or a subgraph of a hexagonal lattice, both of which play critical roles in the previous results on the rigidity of random graphs.

It is important to note that rigidity is inherently a non-local phenomenon: adding a single edge to a graph that has no non-trivial rigid components may rigidify the entire graph (or removing a single edge may cause a large rigid component to shatter). It is this property of rigidity that distinguishes it from the well-studied k-core problem in random graph theory.

In the proof of Theorem 1, we formalize the experimental observation that rigid components, once they appear, are very likely to grow rapidly. Although the proof of Theorem 1 relies mainly on standard tools for bounding sums of independent random variables, our result seems to be the first that directly analyzes rigidity properties of $\mathbb{G}(n, p)$, rather than reducing to a connectivity property.

Related work.

Rigidity: Jackson, et al. [6] studied the space of random 4-regular graphs and showed that they are almost

surely globally rigid. The approach in [6] is based on combining results on the linking rigidity and connectivity proved in [6] and also [5, 12] with connectivity properties of 4-regular random graphs. These techniques rely in an essential way on the 4-regular model; in the $\mathbb{G}(n, p)$ model they are able to prove that when $p = n^{-1}(\log n + 2\log \log n + \omega(1))$, $\mathbb{G}(n, p)$ is almost surely rigid, but have no results for $\mathbb{G}(n, p)$ when the expected number of edges is $O(n)$. The proof of the $\mathbb{G}(n, p)$ result from [6] relies on the existence of a very large 6-core, so it cannot be easily adapted to our setting when c is close to 4 (below the threshold for even the 4-core to emerge [15]).

Percolation: Holroyd [4] extended the formal study of connectivity percolation to rigidity percolation in the hexagonal lattice. He shows, via a reduction to connectivity percolation, that there is an edge-probability threshold for the existence of an infinite* rigid component in the hexagonal lattice which is higher than that for connectivity. It is also shown in [4] that the infinite rigid component, when it exists, is unique for all but a countable set of edge probabilities p. All the proofs in [4] rely in an essential way on the structure of the hexagonal lattice (in particular that a suitably defined tree in its dual graph is a dual of a rigid component).

Random graphs: The fundamental k-core problem in random graph theory has been studied extensively, with a number of complete solutions. Łuczak [13] first proved that for $k \geq 3$, the (it is always unique, if present) k-core is, almost surely, either empty or has linear size. Pittel, et al. solved the k-core problem, giving an exact threshold for its emergence and bounds on its size [15]. Janson and Luczak gave an alternative proof of this result, using simpler stochastic processes [9]. All these results are based on analyzing a process that removes low-degree vertices one at a time, which does not apply in the rigidity setting.

2 Preliminaries

In this section we give the technical preliminaries required for the proof of Theorem 1.

Combinatorial rigidity

An **abstract bar-and-joint framework** (G, ℓ) is a graph $G = (V, E)$ and vector of non-negative **edge lengths** $\ell = \ell_{ij}$, for each edge $ij \in E$. A realization $G(\mathbf{p})$ of the abstract framework (G, ℓ) is an embedding of G onto the planar point set $\mathbf{p} = (\mathbf{p}_i)_1^n$ with the property that for all edges $ij \in E$, $||\mathbf{p}_i - \mathbf{p}_j|| = \ell_{ij}$. The framework (G, ℓ) is **rigid** if it has only a discrete set of realizations modulo trivial plane motions, and is **flexible** otherwise.

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*Rigidity of infinite frameworks is a subtle concept, and [4] devotes careful attention to its development.

A graph $G = (V, E)$ is $(2, 3)$-**sparse** if every subgraph induced by $n' \geq 2$ vertices has at most $2n' - 3$ edges. If, in addition, G has $2n - 3$ edges, G is $(2, 3)$-**tight** (shortly, Laman).

The Maxwell-Laman Theorem completely characterizes the rigidity of generic planar bar-joint frameworks.

Proposition 2 (**Maxwell-Laman Theorem** [10]). *A generic bar-joint framework in the plane is minimally rigid if and only if its graph is $(2, 3)$-tight.*

Genericity is a subtle concept, and we refer the reader to our paper [16] for a detailed discussion. Here, all we need to know is that almost all $\mathbf{p}$ are generic, and that all generic frameworks $G(\mathbf{p})$ have the same rigidity properties.

If G contains a spanning Laman graph it is $(2, 3)$-**spanning** (shortly **rigid**). A rigid induced subgraph is called a **rigid block** (shortly **block**), and an inclusion-wise maximal block is a **rigid component**. By [11, Theorem 5], every graph decomposes uniquely into rigid components, and every edge is spanned by exactly one rigid component. A rigid component is non-trivial if it contains more than one edge. Figure 1(a) shows and example of a Laman graphs. Figure 1(b) has an example of a flexible graph with its rigid components indicated; they are the two triangles and two trivial rigid components consisting of a single edge only.
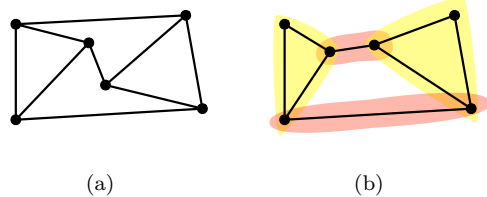

(a)   (b)

Figure 1: Laman graphs and rigid components: (a) a Laman graph on $n = 6$ vertices; (b) a flexible graph with its rigid components indicated.

We summarize the properties of rigid graphs and components that we will use below in the following proposition.

Proposition 3 (**Properties of rigid graphs and rigid components**). *Let $G = (V, E)$ be a simple graph with n vertices: (a) G decomposes uniquely into rigid components, and every edge is in some component [11, Theorem 5]; (b) Adding an edge to a graph G never decreases the size of any rigid component [11, Theorem 2]; (c) If G' is a block in G with vertices $V' \subset V$ and there is a vertex $i \notin V'$ with at least two neighbors in V', then G' is not a rigid component of G; (d) If G has at least $2n - 2$ edges, then it contains a component spanning at least 4 vertices [11, Theorem 2 and Theorem 5].*

What we have presented here is a small part of a well-developed combinatorial and algorithmic theory of (k, ℓ)-sparse graphs. We refer the reader to [11] for a detailed treatment of the rich properties of sparse graphs.

The density lemma

A very useful tool is the following result on the size of dense subgraphs in $\mathbb{G}(n, c/n)$ due to Łuczak [13].

Proposition 4 (Density Lemma [13]). *Let a and c be real constants with $a > 1$ and $c > a$. There is a constant $t(a, c) > 0$, not depending on n, such that any subgraph of $\mathbb{G}(n, c/n)$ with n' vertices inducing at least an an' edges has $n' \geq t(a, c)n$, almost surely.*

3 Proofs

In this section we prove the main result of this paper.

Theorem 1 (Size and emergence of a large rigid component). *Let $c > 0$ be a constant. Almost surely, all rigid components in $\mathbb{G}(n, c/n)$ span 2, 3, or $\Omega(n)$ vertices. If $c > 4$, then almost surely there is a rigid component of size at least $n/10$.*

Proof outline.

Here is the proof strategy in a nutshell. Because any rigid component with $n' \geq 4$ vertices must be somewhat dense, the very general bound of Proposition 4 implies that for $p = c/n$ all the rigid components are either trivial, triangles, or spanning a constant fraction of the vertices in $\mathbb{G}(n, c/n)$ (Lemma 5). We then improve upon our bounds on the probability of rigid components of size sn, for $s \in (0, 1)$ by formalizing the observation that such rigid components are likely to "grow" (Lemma 7) and then optimizing s (Lemma 8).

The rest of this section contains the details.

Rigid components have either constant or linear size

We start by proving that non-trivial rigid components are all very large or triangles, almost surely.

Lemma 5. *Let $c > 0$ be a fixed constant. Almost surely, all rigid components in $\mathbb{G}(n, c/n)$ have size 2, 3, or $\Omega(n)$.*

Proof. By Proposition 3(a), any rigid component on $n' \geq 4$ vertices spans at least $\frac{5}{4}n'$ edges (with equality for $n' = 4$). The lemma then follows from Proposition 4 (with $a = \frac{5}{4}$) and the well-known fact that almost surely $\mathbb{G}(n, c/n)$ contains a triangle [2, Theorem 4.1, p. 79]. $\square$

For $c > 4$, the number of edges in $\mathbb{G}(n, c/n)$ implies that it has at least one large rigid component, almost surely.

Lemma 6. *Let $c > 4$. Almost surely, $\mathbb{G}(n, c/n)$ contains at least one rigid component of size $\Omega(n)$.*

Proof. For any $\epsilon > 0$ $\mathbb{G}(n, (4 + \epsilon)/n)$ has at least $2n - 2$ edges with high probability. Proposition 3(d) then implies that almost surely $\mathbb{G}(n, (4 + \epsilon)/n)$ contains at least one rigid component with at least 4 vertices. By Lemma 5, all of these span at least $t(a, 4 + \epsilon)n$ vertices.

By Proposition 3(b) the size of rigid components is an increasing property and [2, Theorem 2.1, p. 36], this lower bound on size holds, almost surely, for any $c > 4$. $\square$

For $c > 4$ the largest component is very large

We now turn to improving the lower bound on the size of rigid components. To do this, we will use the maximality of rigid components as well as their edge density. In what follows $\mathrm{Bin}(N, p)$ denotes a binomial random variable with parameters N and p; in particular, the number of edges induced by n' vertices in $\mathbb{G}(n, p)$ has distribution $\mathrm{Bin}\left(\binom{n'}{2}, c/n\right)$.

Lemma 7. *The probability that a fixed set of k vertices spans a rigid component in $\mathbb{G}(n, c/n)$ is at most*

$$\mathbf{Pr}\left[\mathrm{Bin}\left(k^2/2, c/n\right) \geq 2k - 3\right] \cdot$$
$$\left((1 - c/n)^k + k\frac{c}{n}(1 - c/n)^{k-1}\right)^{n-k} \quad (1)$$

Proof. To induce a rigid component, a set V' of k vertices must span at least $2k - 3$ edges by Proposition 3(a). By Proposition 3(c) if V' spans a rigid component, no vertex outside of V' can have more than one neighbor in V'. The two terms in (1) correspond to these two events, which are independent. $\square$

Remark: This estimate of the probability of a set of vertices inducing a rigid component is very weak, since it uses only the number of edges induced by V' (not their distribution) and the simplest local obstacle to maximality. Any improvement in this part of the argument would translate into improvements in the lower bound on the size of rigid components.

Lemma 8. *For $c > 4$, almost surely all rigid components span at least $n/10$ vertices.*

Proof. With the assumptions of the lemma, by Lemma 6, $\mathbb{G}(n, c/n)$ almost surely has no blocks of size smaller than tn, where t is a constant independent of n. It follows from Proposition 3(a) that $\mathbb{G}(n, c/n)$ almost surely has no rigid components smaller than tn.

Let X_k to be the number of rigid components of size k. We will show that $\mathbf{E}[X_4] + \cdots + \mathbf{E}[X_{n/10}] = o(1)$, which implies the lemma by a Markov's inequality. As noted

above, $\mathbf{E}[X_4] + \cdots + \mathbf{E}[X_{tn}] = o(1)$, so we concentrate on terms where $k \in [tn, n/10]$. By Lemma 7:

$$\mathbf{E}[X_k] \leq \binom{n}{k} \mathbf{Pr}\left[\text{Bin}\left(k^2/2, c/n\right) \geq 2k - 3\right] \cdot$$
$$\left(\left(1 - c/n\right)^k + k\frac{c}{n}(1 - c/n)^{k-1}\right)^{n-k} \quad (2)$$

Setting $k = sn$ and letting c approach 4, we can use a Chernoff bound [14, Theorem 4.1, p. 68] to bound the right-hand-side of (2) by $e^{-\Theta(n)}$ when $s \in [t, 1/10]$. This then implies that $\sum_{k=tn}^{n/10} \mathbf{E}[X_k] \leq ne^{-\Theta(n)} = o(1)$.

By Proposition 3(b) the size of rigid components is an increasing property, so [2, Theorem 2.1, p. 36], this lower bound on size holds almost surely for any $c > 4$. □

4   Conclusions and open problems

We considered the question of the size and emergence of rigid components in a random graph $\mathbb{G}(n, c/n)$ as c increases, and we proved that almost surely all rigid components in $\mathbb{G}(n, c/n)$ are single edges, triangles or span $\Omega(n)$ vertices. For $c > 4$, we proved that, almost surely, the largest rigid components span at least $n/10$ vertices.

The most natural open question is whether there is a threshold constant for rigid components in $\mathbb{G}(n, p)$.

Question 9 (**Existence of a threshold constant**). *Is there a constant c_r at which a linear-sized rigid component appears in $\mathbb{G}(n, (c_r + \epsilon)/n)$ almost surely, and $\mathbb{G}(n, (c_r - \epsilon)/n)$ almost surely has no large rigid components?*

The other important question is about the structure of large rigid components when they emerge.

Question 10 (**Structure of large rigid components in $\mathbb{G}(n, c/n)$**). *Is there almost surely only one large rigid component in $\mathbb{G}(n, c/n)$, and what are the precise bounds on its size?*

We have observed in computer simulations that when linear sized rigid components are present, there is only one, and it is much larger than $n/10$.

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