On the Height of a Homotopy

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Abstract

Given 2 homotopic curves in a topological space, there are several ways to measure similarity between the curves, including Hausdorff distance and Fréchet distance. In this paper, we examine a different measure of similarity which considers the family of curves represented in the homotopy between the curves, and measures the longest such curve, known as the *height* of the homotopy. In other words, if we have two homotopic curves on a surface and view a homotopy as a way to morph one curve into the other, we wish to find the longest intermediate curve along the morphing.

In this paper, our model assumes we are given a pair of disjoint embedded homotopic curves (where the endpoints remained fixed over the course of the homotopy) in an edge-weighted planar triangulation satisfying the triangle inequality. We prove that among minimal height homotopies between the two curves, there exists an embedded isotopy; in other words, the homotopy with minimum height never makes a "backwards" move and results in disjoint simple intermediate curves.

1 Introduction

There are many ways of measuring similarity between curves. Hausdorff distance is one common measure, which is (intuitively) the maximum distance that an adversary can force by picking a point on one curve and allowing you to choose any point on the other curve. While Hausdorff distance does measure closeness in space, it does not take into account the flow of the curve in space; two curves may have small Hausdorff distance but still not be "similar".

A second metric for measuring similarity between curves in Euclidean space is the Fréchet distance, which is the minimum length of a leash required to connect a man and dog as they travel, from one endpoint to the other, without backtracking, along the two curves. Fréchet distance is used in different applications as a more accurate measure of similarity, and algorithms have been developed to compute Fréchet distance in several different settings [1, 8, 9]. Several variants, such as geodesic Fréchet distance [3] and homotopic Fréchet distance [2], have also been introduced to generalize the notion of Fréchet distance to more general settings.

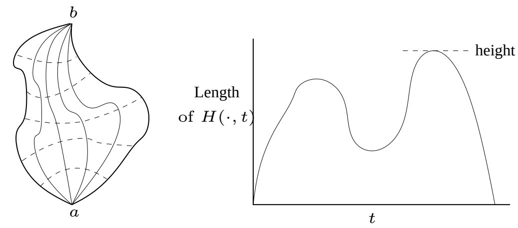

Figure 1: The height of the homotopy measures the maximum length of the solid curves "parallel" to α and β, while Fréchet distance measures the maximum length of the dashed "transverse" curves.

In this paper, we examine a metric for measuring similarity between curves which is in many ways orthogonal to standard Fréchet distance. Any homotopy $H : I \times I \to S$ between two curves yields two families of curves: one set $H(s_0, t)$ (for fixed s_0) that run "between" the two curves being examined and the other $H(s, t_0)$ (for fixed t_0) that run "parallel" to the the curves being examined, see Figure 1. Fréchet distance is the maximum length curve in the first family of curves, $H(s, \cdot)$, while the *height* of the homotopy is the maximum length curve in the second family, $H(\cdot, t)$.

We fix our model as a planar triangulation with weighted edges satisfying the triangle inequality, where the two input curves are constrained to lie along the boundary of the graph. However, it is worth noting that all proofs generalize to graphs embedded on surfaces, where edges in the graph satisfy the triangle inequality and the input curves are homotopic cycles; details of those generalizations are omitted due to space, but will appear in future work.

Borrowing the concept of *thin position* from 3-manifold topology, we will show that among the minimal height homotopies between disjoint paths there is one that never "reverses direction" or "collides with itself". Thin position was developed by Gabai [6] and used by Thompson in the 3-sphere recognition algorithm [10]. The technique focuses on studying local properties of a sequence and then using local optimality conditions to prove global properties. We use this concept to prove the main theorem of the paper and also provide a characterization of minimal "complexity" move sequences.

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2 Definitions

We will be working on a triangulated surface M, where curves lie along the edges of the triangulation and the edges of the triangulation are weighted and satisfy the triangle inequality. In general, a *path* is a continuous map $p : [0, 1] \to M$. However, we restrict paths to follow the edges of the triangulation, where each edge is oriented consistently with traversing starting at $p(0)$ and ending at $p(1)$. The *length* of a path p, written $|p|$, is the sum of the weights of the edges (with multiplicity) in the path.

A path is a *geodesic* if it is impossible to perform a local reduction in its length. In other words (since the underlying graph is unweighted), a path is a geodesic if no edge in the path is immediately followed by its reversal and if no two cofacial edges appear consecutively along the path. Note that this is not the same as being a shortest path, as it is a purely local condition.

A path on a surface is *simple* or *embedded* if it is 1-1. Since we will be restricting paths to lie along edges of a graph on the surface, the same edge or vertex may appear many times in a path. Because of this, we will examine paths that have been perturbed in an infinitesimally small neighborhood of the edges of the triangulation. We will say a path is *simple* if there exists such a perturbation to an embedded curve. Likewise, two paths will be considered *disjoint* if, after an infinitesimal perturbation, they have no points in common.

Two curves $\gamma_1, \gamma_2 : [0, 1] \to M$ are *homotopic* if there is a continuous map $H : [0, 1] \times [0, 1] \to M$ such that $H(0, t) = \gamma_1(t)$ and $H(1, t) = \gamma_2(t)$. In other words, two curves are homotopic if you can continuously deform one to the other. We say the two curves are *isotopic* if for each fixed t and $x \in [0, 1]$, each curve $H(x, t)$ is a homeomorphism, or both onto and 1-1. This is much stronger than simple homotopy, since it insures the the continuous deformation consists of simple curves.

However, in our setting, paths are restricted to edges of the triangulation, so the continuous deformation required for homotopy and isotopy are not well defined. We will use an alternate mechanism to move from one path to another. We will study a *move sequence* from one path to another where each move is one of the following *elementary moves*.

- *Face lengthening* : A move from a single edge e_0 across a face to two edges e_1 and e_2.
- *Face shortening* : A move from two consecutive cofacial edges e_0 and e_1 across a face to a single edge e_2.
- *Spike* : Move across a single edge, so that an edge e followed by its reversal is included in the new path.
- *Reverse spike* : A reverse spike move, where an edge and its immediate reversal is removed from the path.

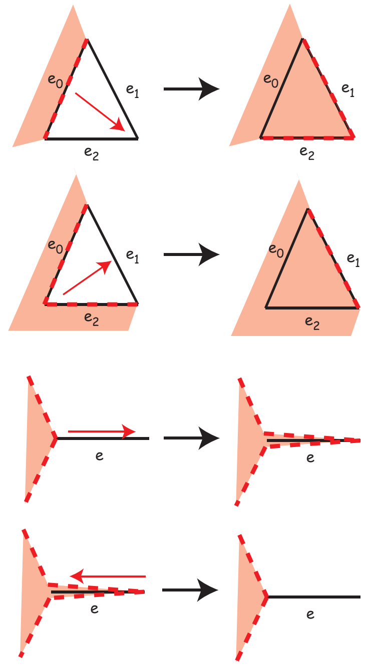

Figure 2: Elementary moves (top to bottom): face lengthening, face shortening, spike and reverse spike.

We will refer to the set of paths obtained by applying elementary moves one at a time as *intermediate paths.*

We can connect a move sequence with k moves to an obvious implied homotopy $H : [0, 1] \to M$, where $H(s, 1/i)$ is equal to the i^{th} intermediate path in the sequence; the homotopy remains fixed between these paths except where the elementary move is being performed. If each of these intermediate paths is simple, then we will refer to the move sequence as *simple*. This is equivalent to saying that the implied homotopy can be perturbed to be an isotopy.

We would like to be able to say that, under appropriate conditions, move sequences never backtrack and proceed monotonically from one path to another. To be precise, consider a transverse orientation on a path that (locally) indicates where the path was previously. A move is considered *(locally) forward* if the move respects the transverse orientation. Figure 3 shows a forward move applied to a path where the transverse orientation is represented by shading; here, the shading is "behind" the curve, so the forward move goes away from the shaded side. Note that since this is purely local, a forward move may still cause the intermediate path to be non-simple. Also, note that move sequences consisting of only locally forward moves can have "spirals." We define a move sequence to be *embedded* if it is simple and only uses forward moves. This is equivalent to saying that after a perturbation, its associated homotopy is an isotopy that is an embedding everywhere except at the preimage of the two endpoints of the paths. Es-

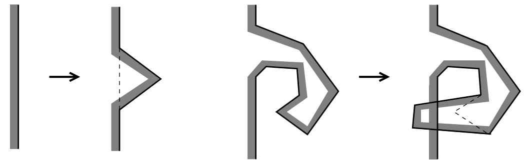

Figure 3: Two forward moves. Locally both move away from the region previous visited. Globally, the second results in a non-simple path.

sentially, an embedded move sequence has intermediate paths that move smoothly across the disk, never crossing themselves or other intermediate paths, so that the situation on the right of Figure 3 can never happen.

The *height* of a homotopy is the maximum length of any intermediate curve: $\max_{t\in[0,1]} |H(\cdot,t)|$; similarly, the *height* of a move sequence is the length of the longest curve in the sequence. We wish to determine the minimum height homotopy between two curves which form the boundary of a planar, unweighted triangulation; in other words, we want the morphing between these two curves that keeps the maximum length of an intermediate curve as small as possible. However, it is not immediately obvious that this homotopy is embedded or forward; our main result, stated formally and proven in the next section, is that some minimum height move sequence is embedded and proceeds uniformly from one path to another without spirals or other degeneracies.

To accomplish this, we need a more precise way to compare two move sequences. Given a sequence of moves, the *length spectrum* is the set of all lengths of the intermediate paths in the sequence. Two length spectrums can be compared by ordering each in decreasing order and comparing the two lists lexicographically. A move sequence is said to be in *thin position* if its length spectrum is lexicographically minimal among all possible move sequences between the same paths. A move sequence that is in thin position has minimal height. Furthermore, every subsequence of moves also has minimal height.

A move sequence is *locally thin* if you cannot decrease the lengths in its length spectrum by any of the following local improvements:

1. Remove a pair of sequential moves where the intermediate curves before and after the pair of moves are combinatorially identical.

2. Reverse the order of a path lengthening move followed by a path shortening move that are independent of each other.

3. Replace a pair of moves that accomplish the result as a single move. For example, a spike move followed by an adjacent face shortening move can be replaced by a single face lengthening move.

We will see that embedded locally thin move sequences share many properties with move sequences that are in thin position.

3 Weighted Planar Triangulations

The setting for all our results in the next two sections is a planar, weighted triangulation (so our underlying manifold is a disk) where the edges weights satisfy the triangle inequality, with two distinguished vertices a and b on the outer face of the graph. Our goal is to characterize the minimum height homotopy from one side of the outer face (a path from a to b along the outer face) to the other side of the outer face.

At each stage of a homotopy from one boundary curve to the other, we have a connected curve between a and b. Our goal is to argue that in a minimum height homotopy, these intermediate paths never move backwards - namely, once an elementary move occurs, it will never be in our interest to move back across that face or edge. We will show that any move sequence that contains a backwards move is not in thin position (which immediately implies that it cannot be a minimum height homotopy). Furthermore, the move sequence must be embedded or, equivalently, the homotopy induced by the move sequence can be infinitesimally perturbed to be an embedded isotopy.

Theorem 1 *Given a move sequence from one side of the boundary of an unweighted planar triangulation to the other side, if the move sequence is in thin position, then it is embedded.*

Corollary 2 *There exists a minimum height moves sequences that is embedded.*

The proof of Theorem 1 will follow from the following two propositions. The first shows that there are no backwards moves, and the second shows that any move sequence consisting of only forward moves is embedded.

Proposition 3 *Any move sequence in thin position will never contain a backwards move.*

The proof of this proposition relies on the observation that if there is a backwards move for a move sequence in thin position, then the move immediately prior to it must share an edge with the backwards move. In a case by case analysis, this pair of moves can be replaced by different moves that reduce the complexity of the move sequence.

Proposition 4 *A forward move sequence from an arc on the boundary of a disk to the complementary arc in the boundary is embedded.*

This can be proved using arguments involving covers of topological spaces.

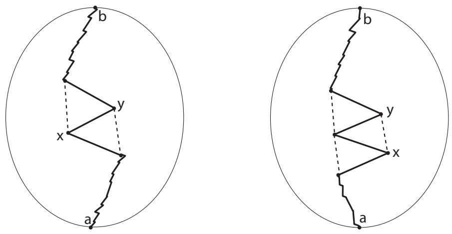

Figure 4: The configurations of paths at the local maxima of a move sequence in (locally) thin position. The paths from a to x and y to b are geodesics.

4 Characterizing Move Sequences in Thin Position

In applications of thin position to 3-manifold topology, the local maxima and minima of sequences that are in thin position have particularly nice properties. The same is true for move sequences.

Theorem 5 *If a move sequence is either in thin position or is embedded and locally thin then:*

1. *A path in the move sequence whose length is a local minimum is a geodesic.*
2. *A path in the move sequence whose length is a local maximum is geodesic everywhere except two or three points, and at these points the path has, up to symmetry, one of the configurations shown in Figure 4.*

5 Extensions and Open Questions

The same case analysis used in the proof Proposition 3 plus a few additional arguments can be used to prove similar results about move sequence between the boundary components of a triangulated annulus; this naturally gives a useful characterization of the minimum height of a homotopy between two curves on a surface.

In Sections 3 and 4, we have characterized the movement of any minimum height homotopy. The primary remaining open question, of course, is to find a polynomial time algorithm which, given two cycles on a combinatorial surface, computes a homotopy of minimum height (or at least the height of the minimum homotopy). Some initial work in this area has been done for the planar version of the problem, where the graph itself is a series parallel graph whose edges do not need to satisfy the triangle inequality [5]. One possible strategy for an algorithm in more general settings would rely on proving that the shortest path appears in a move sequence in thin position, and then recursively computing the minimum height homotopy in each half of the graph using our characterization of local minimum and maximum intermediate paths. However, our proofs do not give that the shortest path will appear in the minimum height homotopy, although we conjecture that it does.

It is not clear that the problem is not NP-Complete, since it bears a close resemblance to finding the cut width of the dual graph. In fact, if we disallow spike moves, the problem becomes equivalent to finding the cut width of the dual graph, which is NP-Hard even in planar graphs. (See [4] for a survey of cut width and similar graph layout problems.)

Acknowledgements

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References

[1] H. Alt and M. Godau. Computing the Fréchet distance between two polygonal curves. *International Journal of Computational Geometry and Applications*, 5(1–2):75–91, 1995.

[2] E. W. Chambers, É. C. de Verdière, J. Erickson, S. Lazard, F. Lazarus, and S. Thite. Homotopic fréchet distance between curves, or walking your dog in the woods in polynomial time. *Computational Geometry: Theory and Applications*, 2009. To appear.

[3] A. Cook and C. Wenk. Geodesic Fréchet and Hausdorff distance inside a simple polygon. Tech. Rep. CS-TR-2007-004, University of Texas at San Antonio, 2007.

[4] J. Díaz, J. Petit, and M. Serna. A survey of graph layout problems. *ACM Comput. Surv.*, 34(3):313–356, 2002.

[5] D. Eppstein. Personal communication, 2008.

[6] D. Gabai. Foliations and the topology of 3-manifolds III. *Journal of Differential Geometry*, 26:479–536, 1987.

[7] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2001.

[8] A. Maheshwari and J. Yi. On computing Fréchet distance of two paths on a convex polyhedron. In *Proceedings of the 21st European Workshop on Computational Geometry*, pages 41–44, 2005.

[9] E. Sriraghavendra, K. Karthik, and C. Bhattacharyya. Fréchet distance based approach for searching online handwritten documents. In *ICDAR07*, pages 461–465, 2007.

[10] A. Thompson. Thin position and the recognition problem for S^3. *Mathematical Research Letters*, 1:613–630, 1994.