

General minisum circle location -extended abstract-

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Abstract

In our paper we approximate a set of given points by a general circle. More precisely, we consider the problem of locating and scaling the unit ball of some given norm k_1 with respect to fixed points on the plane such that the sum of weighted distances between the circle and the fixed points is minimized, where the distance is measured by a norm k_2 . We present results for the general case. In the case that k_1 and k_2 are both block norms, we are able to identify a finite dominating set in \mathbb{R}^3 for the problem, which can be obtained as the intersection of cones.

1 Introduction

Up to now, circle location problems have only been considered using the Euclidean norm and recently for the case of an arbitrary norm k , see [2, 3, 7]. In this paper we generalize the Euclidean view in two respects: First, we locate a convex, symmetric set (i.e. the scaled unit circle with respect to an arbitrary norm k_1) instead of the unit circle with respect to the Euclidean norm ℓ_2 . Second, we measure the distance from points to the circle in a (maybe different) norm k_2 instead of using the Euclidean distance.

Circle locations problems are useful models in operations research. They may be used for the out-of-roundness problem; see Drezner et al. [2]. Other applications include the design of circular public transportation networks (PTN for short). Circular PTNs are very common in practice, e.g., in London, Moscow, Berlin, Hamburg, and Tokyo circular underground or suburban railways can be found. Our model is suited to determine a rough route of a new circular PTN that minimizes the average distance from the customers to the PTN. In a subsequent detailed planning, this tentative route can be adapted to local realities (e.g. buildings, watercourses, parks, and nature protection areas). Besides PTNs, also ring roads are of practical interest; see Pearce [4] and Suzuki [5].

The remainder of the paper is structured as follows: In Section 2 we introduce the notation needed and introduce the problem formulation. In Section 3 we give some results that are valid for any combination of norms k_1 and k_2 . Basically, we show convexity and concavity, respectively, on certain regions for the case where either the center or the radius of the circle is fixed. After that general section we turn towards the case where k_1 and k_2 are both block norms. The main result of this section is the identification of a finite dominating set (FDS for short), i.e. a finite point set that contains at least one optimal solution for the problem. It can be obtained as an intersection of some cones. In Section 5 we present a geometrical interpretation of the points in the FDS. The paper is closed by a conclusion in which also ideas for geometric algorithms are discussed.

2 Problem statement and notation

Consider a set \mathcal{A} of n fixed points, $A_j = (a_j, b_j)$ with associated positive weights w_j , for $j = 1, \dots, n$. In order to simplify the presentation of results we assume that $n \geq 5$ and that no triple of the fixed points is collinear. Our goal is to find a (general) circle such that the sum of shortest distances from the circle to the set \mathcal{A} is minimized. To describe our problem, let two norms k_1, k_2 be given. For any norm k we define the distance between two points X and Y as $k(X, Y) = k(Y - X)$. The general circle we want to locate is the scaled unit ball with respect to norm k_1 . It is determined by its center, $X = (x, y)$, and its radius (or scaling factor), r , and is hence given as $C(X, r) = \{Y \in \mathbb{R}^2 : k_1(Y, X) = r\}$. We use the shortcut $C = C(X, r)$. The distance between a circle $C(X, r)$ and a point A is measured through the norm k_2 . It is given as $d(C, A) = \min_{Y \in C} k_2(A, Y)$, where $Y \in C = C(X, r)$ means that $k_1(X, Y) = r$. The problem may be formulated as

$$\min f(C) = \sum_{j=1}^n w_j d(C, A_j) \quad (\text{GP})$$

As mentioned in the introduction, cones play a central role in our paper. Whenever we refer to a cone we reference to an affine and convex cone that includes its apex.

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3 Results for general norms

In this section we study properties of (GP) which are valid for all norms k_1 and k_2 .

Lemma 1 *The optimal solution of problem (GP) must have a positive radius.*

Lemma 1 ensures that a point ($r = 0$) can never be an optimal solution of the general circle location problem (GP) with more than one existing facility. Also the other extreme $r = \infty$ is only possible if k_1 is neither a smooth norm (i.e. k_1 is differentiable everywhere except at the origin) nor a block norm (see Definition 1):

Lemma 2 *Let k_1 be a smooth or block norm. Then there always exists an optimal solution to (GP) with finite radius $r < \infty$.*

We now investigate the convexity properties of the function $h(r) = d(C(X, r), A)$ as a function of the radius r only, i.e. with center X and A being fixed. We obtain the following result:

Lemma 3 *Given A and X as two points in the plane, the function $h(r) = d(C(X, r), A)$ is convex for all $r \leq k_1(X, A)$ and concave for all $r \geq k_1(X, A)$.*

For the case where the radius is fixed and we are looking for a center point X , we are able to show similar results:

Lemma 4 *The function $h(X) = d(C(X, r), A)$ is quasiconcave, if $X \in (A, r, \leq) = \{X \in \mathbb{R}^2 : k_1(X, A) \leq r\}$, and quasiconvex if $X \notin (A, r, \leq)$.*

Furthermore, it can be shown that $h(X)$ attains its global minimum for all $X \in (A, r)$ and has a local maximum in $X = A$.

4 The case of block norms

In this section we analyze the case in which both norms k_1 and k_2 are block norms. A common definition of block norms is the following:

Definition 1 (see [8]) *Let B be a compact convex polyhedron in \mathbb{R}^2 that is point symmetric with respect to the origin O . The **block norm** with respect to B is then defined as*

$$k_B(X) := \inf\{\lambda > 0 : X \in \lambda B\}.$$

The extreme points $\text{Ext}(B) = \{E_1, \dots, E_L\}$ of B are called **fundamental directions** of the block norm k_B . B is also referred to as the unit ball of k_B .

Throughout this paper we assume that the extreme points $\text{Ext}(B)$ of any block norm are numbered in counterclockwise order, i.e. we have

$$\text{Ext}(B) = \{E_1, E_2, \dots, E_G, -E_1, -E_2, \dots, -E_G\}.$$

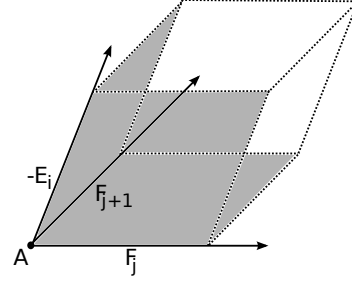


Figure 1: Cone $N_A(i, j)$. Dashed lines: reference lines to point out the dimensional shape of the cone. Facets in dark gray: part of the boundary of the cone.

In the following let k_1 and k_2 be block norms with respect to the polyhedrons B_1 and B_2 , respectively. For the sake of simple distinction between fundamental directions of k_1 and k_2 we identify directions of k_1 by character E and directions of k_2 by character F , i.e. $\text{Ext}(B_1) = \{E_i : i = 1, \dots, L\}$ and $\text{Ext}(B_2) = \{F_i : i = 1, \dots, G\}$.

The main idea of this section is the following: The distance $d(C_1, A)$ between a point A and a k_1 -circle $C_1 = C(X, r)$ can be interpreted as the radius s of the smallest k_2 -circle $C_2 = C(A, s)$ that touches C_1 . Since two polyhedrons intersect always at a vertex of one of them, we can distinguish the following cases.

Case 1: C_2 touches C_1 in a vertex of C_1 . Let $E \in \text{Ext}(B_1)$ be the fundamental direction of k_1 that defines this vertex of C_1 . Then the distance from A to C_1 is given as $k_2(X + rE, A)$.

Case 2: C_2 touches C_1 in a vertex of C_2 . Let $F \in \text{Ext}(B_2)$ be the corresponding fundamental direction of k_2 . Then the distance from A to C_1 is given as $\min\{|\lambda| : A + \lambda F \in C_1\}$.

Distinguish these two cases, we are able to identify a cell tessellation of \mathbb{R}^3 such that $d(C(X, r), A)$ is concave in (X, r) .

Vertices of C_1

In the following we will investigate Case 1.

Definition 2 *Given point $A \in \mathbb{R}^2$ and a fundamental direction $E \in \text{Ext}(B_1)$ of k_1 let $g_{A,E} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, $g_{A,E}(X, r) = k_2(X + rE, A)$.*

Given $A, U, V, W \in \mathbb{R}^2$ we denote the cone

$$\left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} U \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} V \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} W \\ 0 \end{pmatrix} : \alpha_i \geq 0 \right\}$$

by $N(A, U, V, W)$. We use the notation $N_A(i, j)$ to denote the cone $N(A, -E_i, F_j, F_{j+1})$, where $E_i \in \text{Ext}(B_1)$ and $F_j, F_{j+1} \in \text{Ext}(B_2)$; see Figure 1 for a sketch of a cone $N_A(i, j)$. Notice that for any $(X, r) \in N_A(i, j)$

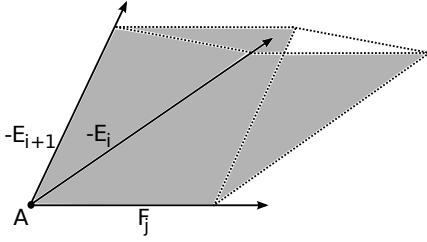


Figure 2: Cone $M_A(i, j)$. Dashed lines: reference lines to point out the dimensional shape of the cone. Facets in dark gray: part of the boundary of the cone.

we have $g_{A, E_i}(X, r) = k_2(\alpha_1 F_j + \alpha_2 F_{j+1})$. As a consequence of elementary block norm properties we obtain:

Lemma 5 $g_{A, E_i}(X, r)$ is affine linear on cone $N_A(i, j)$.

The cones $\{N_A(i, j) : i = 1, \dots, L, j = 1, \dots, G\}$ define a decomposition of $\mathbb{R}^2 \times [0, \infty)$ into polytopes. Let \mathcal{P} denote the set that contains these polytopes. Due to Lemma 5 we obtain immediately:

Corollary 6 The function

$$\mu_A(X, r) := \min_{E \in \text{Ext}(B_1)} g_{A, E}(X, r)$$

is concave on any polytope $P \in \mathcal{P}$.

Note that $\mu_A(X, r)$ is the k_2 -distance between A and the vertex of $C(X, r)$ that is the closest one to A .

Facets of C_1

Now we turn towards Case 2 and consider the facets of a k_1 -circle. Since the fundamental directions of k_1 are numbered in counterclockwise order, any facet \mathcal{S} of a circle $C(X, r)$ is uniquely determined by two neighbored fundamental directions. Hence, for all $E_i \in \text{Ext}(B_1)$ we may define

$$\mathcal{S}_i(C) = \{X + \alpha E_i + \beta E_{i+1} : \alpha + \beta = r, \alpha, \beta \geq 0\},$$

where $E_{L+1} := E_1$. We define distance between facets of $C(X, r)$ and point A :

Definition 3 Given $A \in \mathbb{R}^2$, $E_i \in \text{Ext}(B_1)$, $F_j \in \text{Ext}(B_2)$ let $h_{A, E_i, F_j}(X, r) = \min\{\lambda \geq 0 : A + \lambda F_j \in \mathcal{S}_i(C(X, r))\}$, where $\min \emptyset := \infty$.

Let $M_A(i, j) := \{(X, r) : \exists \lambda \geq 0 : A + \lambda F_j \in \mathcal{S}_i(X, r)\}$, see Figure 2 for a sketch. Due to block norm properties we obtain:

Lemma 7 If F_j and $E_i - E_{i+1}$ are linearly independent, then the function h_{A, E_i, F_j} is affine linear on $M_A(i, j)$.

It can be shown that h_{A, E_i, F_j} is also linear on certain regions in the case where F_j and $E_i - E_{i+1}$ are not linearly independent. But this case is not of interest, since linear dependence of F_j and $E_i - E_{i+1}$ implies that the ray $A + \lambda F_j$ intersects a vertex of C , i.e. this case is already included in Case 1 discussed above.

The set of cones $M_A(i, j)$, $i = 1, \dots, L$, $j = 1, \dots, G$, defines a decomposition of \mathbb{R}^3 into polytopes. Let \mathcal{R} denote this decomposition and let

$$\nu_A(X, r) := \min_{i=1, \dots, L} \min_{j=1, \dots, G} h_{A, E_i, F_j}(X, r).$$

Notice that $\nu_A(X, r)$ is the k_2 -distance between A and the closest facet of $C(X, r)$. From Lemma 7 we may conclude:

Corollary 8 $\nu_A(X, r)$ is concave on any polytope $P \in \mathcal{R}$.

Finite dominating set

Now, let $\mathcal{P} + \mathcal{R}$ be the set of polytopes we obtain by combining tessellation \mathcal{P} and tessellation \mathcal{R} . As a consequence of Corollary 6 and Corollary 8 we obtain the main results of this section:

Theorem 9 The distance $d(C, A)$ is concave on any polytope $P \in \mathcal{P} + \mathcal{R}$.

Let \mathcal{T} denote the set of all vertices of polytopes $P \in \mathcal{P} + \mathcal{R}$. Then we obtain:

Corollary 10 \mathcal{T} is a finite dominating set for problem (GP), i.e. there exists $(X, r) \in \mathcal{T}$ which is an optimal solution.

5 Geometric interpretation

In this section we will investigate the geometrical meaning of the finite dominating set \mathcal{T} we have found in the previous section. To this end we need the fact that cone $M_A(i, j)$ may be written as follows

$$\left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} - \alpha_1 \begin{pmatrix} E_i \\ -1 \end{pmatrix} - \alpha_2 \begin{pmatrix} E_{i+1} \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} F_j \\ 0 \end{pmatrix} : \alpha_i \geq 0 \right\}.$$

Now, we define two different kinds of hyperplanes which are needed for the tessellation:

Construction planes. This are hyperplanes of the type

$$H = \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} E_i \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} F_j \\ 0 \end{pmatrix} : \alpha_i \in \mathbb{R} \right\},$$

where A is a fixed point, $E_i \in \text{Ext}(B_1)$, and $F_j \in \text{Ext}(B_2)$.

Cone planes. Hyperplanes of the type

$$H = \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} E_i \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} E_{i+1} \\ 1 \end{pmatrix} : \alpha_i \in \mathbb{R} \right\},$$

where A is a fixed point, and $E_i, E_{i+1} \in \text{Ext}(B_1)$.

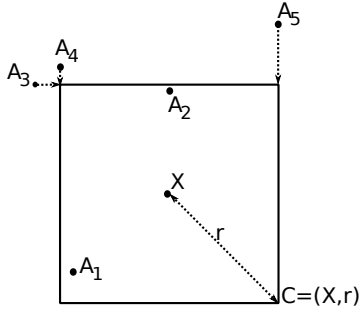


Figure 3: Example with $k_1 = \ell_\infty$, $k_2 = \ell_1$. Optimal solution $C(X, r)$ is the intersection of construction planes through A_3 , A_4 , and A_5 .

Note that any facet of the cones $N_A(i, j)$ and $M_A(i, j)$ are included either in a construction plane or a cone plane or in the plane $\mathbb{R}^2 \times \{0\}$. Hence, any solution $(X, r) \in \mathcal{T}$ can be obtained as the intersection between some of these planes and we have five not necessarily mutually exclusive possible configurations:

1. (X, r) is the intersection of three construction planes. In this case there exist three fundamental directions $F_{j_1}, F_{j_2}, F_{j_3} \in \text{Ext}(B_2)$ and three fixed points A_1, A_2, A_3 such that the ray $A_i + \lambda F_{j_i}$ intersects the circle $C(X, r)$ in a vertex, see Figure 3.

2. (X, r) is the intersection of three cone planes. In this case there exist three fixed points A_1, A_2, A_3 such that $A_i \in C(X, r)$.

3. (X, r) is the intersection of two construction planes and a cone plane. In this case there exist two fundamental directions $F_{j_1}, F_{j_2} \in \text{Ext}(B_2)$ and three fixed points A_1, A_2, A_3 such that the ray $A_i + \lambda F_{j_i}$ intersects the circle $C(X, r)$ in a vertex, $i = 1, 2$, and $A_3 \in C(X, r)$.

4. (X, r) is the intersection of a construction plane and two cone planes. In this case there exist a fundamental direction $F_j \in \text{Ext}(B_2)$ and three fixed points A_1, A_2, A_3 such that the ray $A_1 + \lambda F_j$ intersects the circle $C(X, r)$ in a vertex and $A_i \in C(X, r)$, $i = 2, 3$.

5. $(X, r) \in \{X \in \mathbb{R}^3 : x_3 \leq 0\}$. In this case (X, r) can not be an optimal solution to problem (GP), since any optimal solution must have a positive radius $r > 0$, see Lemma 1.

Although any solution $(X, r) \in \mathcal{T}$ can be represented as the intersection of a construction plane, cone plane or plane $\mathbb{R}^2 \times \{0\}$, the opposite direction is not true, i.e. not any intersection of hyperplanes of these types are a point of \mathcal{T} . Moreover, the descriptions given in (1) to (4) do not uniquely determine a circle. For example there exist uncountably many ℓ_∞ -circles that pass through the three points $A_1 = (0, 0)$, $A_2 = (1, 0)$, and $A_3 = (1, 1)$.

6 Conclusion

We have considered the problem of locating a general circle (i.e. the unit circle with respect to an arbitrary norm k_1) on the plane so as to minimize the sum of weighted distances between some fixed points and the circle, where the distance between points and circle is measured in a maybe different norm k_2 . Primarily, we studied the case where k_1 and k_2 are both block norms. The main result for this case is that there exists a finite set of candidate circles which contains at least one optimal solution for our problem. We have shown, how this set can be obtained by intersections of cones. Furthermore, we explained some geometric properties of these candidate circles.

In this paper we did not discuss algorithmic approaches to solve our problem. A first approach is to compute all intersection points of hyperplanes mentioned in Section 4. Even if we fix the number of fundamental directions of norm k_1 and k_2 this approach costs an exhaustive evaluation of $\mathcal{O}(n^3)$ potential intersection points. A better (output-sensitive) algorithm should be possible that is taking advantage from geometric properties of the cones which generate the candidate set. A possible line of further research may be a sweep line approach on the plane $\{X \in \mathbb{R}^3 : x_3 = 0\}$. It is based on the idea that the relative position of the fixed points contains information about intersection between the FDS generating cones, since the fixed points are the apexes of these cones.

References

- [1] A. Schöbel. Locating Lines and Hyperplanes. Kluwer, Dordrecht, 1999.
- [2] Z. Drezner, G. Steiner, and G.O. Wesolowsky. On the circle closest to a set of points. *Computers and Operations Research*, vol. 29, pp. 637-650, 2002.
- [3] J. Brimberg, H. Juel, and A. Schöbel. Locating a minimum circle in the plane, *Discrete Applied Mathematics*, vol. 157, pp. 901-912, 2009.
- [4] C.E.M. Pearce. Locating concentric ring roads in a city. *Transportation Science*, vol. 8, pp. 142-168, 1974.
- [5] T. Suzuki. Optimal location of orbital routes in a circular city. *ISOLDE X, Sevilla and Islantilla, Spain*, June 2-8, 2005.
- [6] J. Brimberg, H. Juel, and A. Schöbel. Locating a circle on a sphere. *Operations Research*, vol. 55, pp. 782-791, 2007.
- [7] J. Brimberg, H. Juel, M. Körner, and A. Schöbel. Locating a minimum circle on the plane with arbitrary norm. Working paper, 2009.
- [8] J.E. Ward and R.E. Wendell. A new norm for measuring distance which yields linear location problems. *Operations Research*, vol. 28, pp. 836-844, 1980.