Bold Graph Drawings

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Abstract

When a graph is drawn in a classical manner, its vertices are shown as small disks and its edges with a positive width; zero-width edges exist only in theory. Let r denote the radius of the disks that show vertices and w the width of edges. We give a list of conditions that make such a drawing good and that apply to not necessarily planar graphs. We show that if $r < w$, a vertex must have constant degree for a drawing to satisfy the conditions, and if $r \geq w$, a vertex can have any degree. We also give an algorithm that, for a given drawing and a ratio like $r = 2w$, computes the maximum r and w without violating the conditions.

1 Introduction

Possibly the most basic way to draw a graph is to use black, filled disks for the vertices and black, straight line segments that connect two disk centers for the edges. Although edges are usually thought of as having zero-width, to be able to see them they must have at least some positive width. In most cases this width is the same for all edges, and the width is smaller than the diameter of the disks that represent vertices.

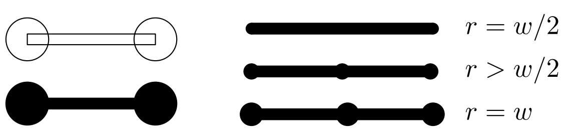

Figure 1: Left, an edge is drawn as a rectangle that connects two centers of vertices in the natural way. Right, a graph with three vertices and two edges, and different ratios of r and w.

In this paper we adopt this rather geometric view of graph drawings. We assume that vertices are drawn as disks with radius r and edges are drawn as rectangles with width w. Note that we cannot really see the ends of an edge because they overlap the incident vertices, see Figure 1. r and w are constants with $r > w/2$; if $r = w/2$, vertices can be hidden in edges as in Figure 1. It seems that the range $w \leq r \leq 2w$ is reasonable.

Obviously, it is important to be able to see from a graph drawing which graph you are looking at. No ambiguity should be present. But also without ambiguity, drawings can look poor, for instance when the drawing of an edge intersects the drawing of a vertex while the vertex and edge are not incident in the graph, see Figure 2(a). For a non-planar drawing of a graph, the drawings of the edges form black regions that may become so large that they can contain a disk as large as a drawing of a vertex. In Figure 2(b), we cannot distinguish between the graph that contains four matching edges and the same graph that additionally, contains an isolated vertex (which may be invisible in the region of the union of the edges). The same problem can show up in a planar drawing of a graph, where close to a high-degree vertex a large black region can arise, see Figure 2(c). Notice that in the latter two cases, the possible ambiguity is resolved if we make r larger (w.r.t. w), although this may cause other problems.

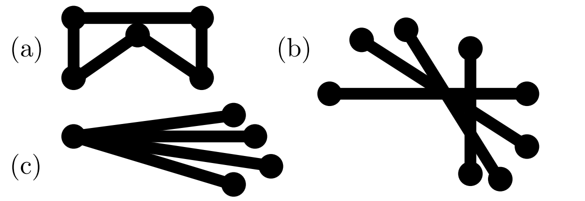

Figure 2: (a) A drawing with an edge and a vertex that intersect without being incident in the graph. (b) and (c) Two drawings of graphs where a black region arises.

Another feature of a good drawing is that it is possible to see at least some part of the boundary of the drawing of each vertex. It may be that from a drawing one can deduce that a vertex must be present even though no part of its boundary can be seen, see Figure 3. Although there is no ambiguity, such drawings are not good.

There has been only little research on graph draw-

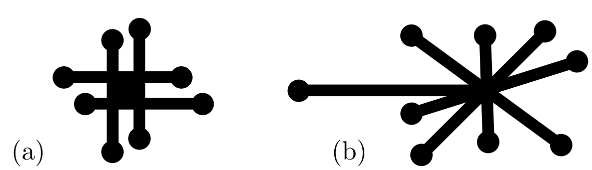

Figure 3: (a) No part of the boundary of the vertex in the middle can be seen, but due to the black region we know it is there. (b) A vertex is in the middle black region because the leftmost edge stops there.

ing related to our setting. Barequet et al. [2] consider drawings of planar graphs where the edge thickness represents a certain quantity or capacity of the edge, and vertices are drawn as large squares. Duncan et al. [5] study graph drawings with edges of a maximum width for planar graphs. Edges are not straight, but are paths between the endpoints which avoid vertices and other paths for edges.

Graph drawing programs allow the standard, straight edge drawings and may have options to draw vertices and edges bold (for example, NEATO of Graphviz [7]).

2 Conditions on good bold graph drawings

Based on the above discussion, we give a list of conditions for any *good* drawing of a graph using width-w rectangles as edges and radius-r disks as vertices:

1. No two vertices intersect.
2. No edge intersects a non-incident vertex.
3. *(Vertex presence)* For every vertex, at least part of its boundary is visible.
4. *(Edge presence)* For every edge, at least part of its boundary is visible.
5. *(Vertex absence)* The region occupied by the union of the edges minus the union of the vertices does not contain any area that can contain a radius-r disk (which could have been an isolated vertex).
6. *(Edge absence)* The region occupied by the union of the edges minus the union of the vertices does not contain any area that can contain a width-w rectangle that could have been an edge between two vertices.
7. There is no point in the plane that is covered by more than two edges, unless those edges are all incident to the same vertex (no face collapse).

With these seven conditions, the arrangement of the drawing will look the way it should, in the sense of no collapsed (white) faces in the complement of the drawing, and the only features that intersect are pairs of edges, pairs of a vertex and an edge that are incident, or multiple edges that share an incident vertex. Figure 4 shows two drawings of a K_7, one where all conditions are satisfied, and one where several are violated.

Notice that the third condition basically disallows an optimal angular resolution at a vertex that has sufficiently high degree (depending on r and w), so a star graph cannot be drawn fully symmetrically. Also notice that the last condition does not allow drawings where there is a triple intersection on zero-width edges, so a K_6 cannot be drawn fully symmetrically where the vertices form a regular hexagon.

We can show that for drawings that satisfy all seven conditions, conditions 4 and 6 are redundant: they are

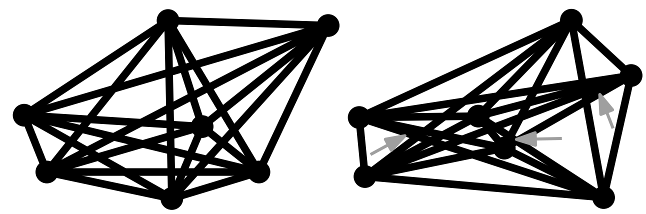

Figure 4: Two drawings of a K_7. Left, one that satisfies the seven conditions. Right, one that violates conditions 2, 5, and 7 (multiple times) at places shown by the grey arrows.

implied by the other five conditions. We need to assume that vertices and edges are *closed* disks and rectangles, that is, they include their boundary.

Lemma 1 *If a drawing satisfies conditions 1, 2, 3, 5, and 7, then it also satisfies conditions 4 and 6.*

Proof. (sketch) For any edge $e = uv$, when it departs from one of its incident vertices, say u, its sides may be obscured by other edges that depart from u. At some point, edge e will not overlap with any such edge any more, otherwise e cannot end at v without violating condition 2. At the point where the boundary of e becomes visible with respect to the other edges that depart from u, there cannot be any other edge containing that point, because then that point would lie on three edges simultaneously, violating condition 7.

If no edges departing from u overlap with e when e leaves the disk of u, then no other edge can intersect e immediately, because then that other edge intersects the disk of u. In both cases, part of the boundary of e is visible, and condition 4 is satisfied. Condition 6 can be shown to hold in a similar manner. □

Assume an embedding of a planar graph is given, with an assignment of coordinates to the vertices. We observe that if the vertices and edges are in non-degenerate position, then there exist positive values of r and w (small enough) that will make all seven conditions hold.

3 Degree of nodes

For two edges e and e' incident to v, we say that they *are together* at distance d if the circle centered at v and with radius d intersects the union of the drawings of e and e' in one connected component. Similarly, more edges incident to v can *be together* at distance d. When edges are together, they define a *diverging angle*, which is the largest angle between the edges that are together.

For a set of edges incident to v, we say that they *come loose* at distance d if they are together at distances $\leq d$, but at distances $> d$, they are not together; the set will be partitioned into two or more non-empty subsets that are together at some distance $d' > d$. The diverging angles of these subsets are smaller than the diverging angle of the original set, and their angular intervals with respect to v do not overlap. We observe that the diverging angle is 0 if and only if an edge is not together with any other edge. If the diverging angle is > 0, then two or more edges must be together, and at some distance they must come loose. An edge can only end at a vertex when it is not together with any other edge, otherwise the drawing will violate condition 2.

Lemma 2 *If $w > r$, then every vertex in a good bold drawing has constant degree. If $w \leq r$, then some vertex in a good bold drawing can have arbitrarily large degree.*

Proof. Notice that both claims are true if and only if they are true for star graphs, so let v be the high-degree node of a star graph.

For the first claim, let $w = r + \delta$ for some $\delta > 0$. Take the two edges e and e' incident to v that make the smallest angle, and denote this angle by α. Let d be the distance where these edges come loose, and let p be the point on e and e' furthest from v, so p is at distance d from the center of v. Then the segment s through

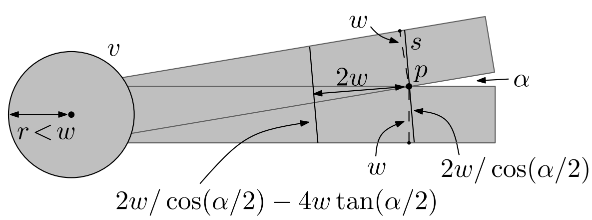

Figure 5: If $r < w$, then v must have constant degree.

p that makes an angle of $\pi/2 - \alpha/2$ with both e and e' and lies inside their union has length $2w/\cos(\alpha/2)$. Consider the isosceles trapezoid that has s as the base, with height $2w$, and which lies inside the union of e and e'. Then the top side has length $2w/\cos(\alpha/2) - 4w\tan(\alpha/2) > 2w(1-\alpha)$. We see that if $\alpha \leq \delta$, then a radius-r disk fits inside the isosceles trapezoid and hence, in the union of e and e'. Hence, α must have at least some constant value $> \delta$ to not violate condition 5, so v must have constant degree.

For the second claim, let $w = r$. Let E be a set of edges incident to v that are together and that have a diverging angle α; assume $0 < \alpha < \pi/4$. We will show that we can construct a drawing where E comes loose at a distance d before a radius-r disk fits inside the union of the drawings of the edges in E, and E is partitioned into two subsets that both have a diverging angle > 0. We can repeat the argument on the two subsets, which shows that E can have arbitrarily many edges.

For the two outer sides of the union of the drawings of the edges in the set E (the ones that determine the diverging angle α), consider a ray s starting at v's center

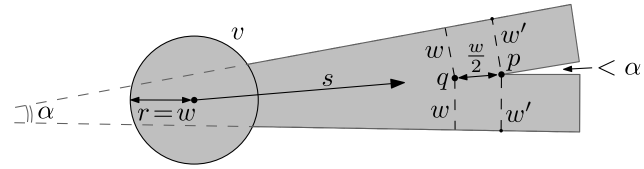

Figure 6: A set of edges that are together with diverging angle $\alpha > 0$ can come loose into two subsets so that both have diverging angle > 0.

whose angle lies in the middle of the outer sides. Let q be the point on s that has distance w to the outer sides, and let p the point on s that is $w/2$ further from v. We will make sure that E comes loose at p, and by construction, no radius-r disk fits inside E before p. Since the distance from p to each of the outer sides is $w' > w$, the diverging angles of the new subsets is strictly greater than 0, and therefore these subsets contain more than one edge each. This finishes the proof. □

4 Optimizing r and w for a given drawing

We next study the problem of computing maximal values of r and w while satisfying the conditions. More precisely, we assume that r/w is a fixed constant that is at least 1, a drawing of a graph is given, and we wish to determine the smallest value of r (and simultaneously of w) that violates at least one of the conditions. From now on, we will only refer to the value of r, since it specifies the value of w.

For any value of r, all faces of the complement of the union of vertices (disks) and edges (rectangles) are bounded by circular arcs and straight edges. We denote the collection of faces by $\mathcal{F}_r$ and its edges—called *sides* to avoid ambiguity—by $\mathcal{S}_r$. If condition 1 is not violated and no vertex is isolated, then every circular side of $\mathcal{S}_r$ is adjacent to two straight sides. The circular sides are concave for the faces of $\mathcal{F}_r$, whereas all endpoints of sides in $\mathcal{S}_r$ are convex.

Let n be the number of vertices and m the number of edges of the graph that is drawn, and assume non-degeneracy. Let r be small enough so that no condition is violated, then we let M denote the number of sides in $\mathcal{S}_r$. There are $O(m)$ circular sides and $O(M)$ straight sides in $\mathcal{S}_r$. We have $m = O(n^2)$ and $M = O(m^2) = O(n^4)$. Since we are interested in the smallest value of r where some condition is violated, we can assume when we analyze some condition that the other conditions are not yet violated. We will take the minimum over the values of r for the first violation of each condition.

The smallest value of r such that condition 1 is violated can easily be computed in $O(n \log n)$ time using a closest pair algorithm.

For condition 2, we can use a brute-force algorithm that runs in $O(nm)$ time, but a more efficient algorithm exists using partition trees [1]. For every edge we can find the closest vertex in its perpendicular strip by preprocessing the vertices into a partition tree with associated structures that can answer tangent queries. This leads to a running time of $O^*(m \cdot n^{1/3} + n^{4/3})$ or $O^*(m + n^2)$, for example (O^*-notation leaves out log-factors). Which is better depends on how much larger m is than n.

We can determine the smallest value of r such that condition 3 is violated in $O(m \log n)$ time (assuming that conditions 1 and 2 are not violated). In fact, violation of condition 3 does not depend on the value of r since r/w is fixed. For any vertex (disk), only the incident edges can cause the boundary of the disk to be invisible, and we can test this by sorting the incident edges by angle.

For condition 7, we compute the faces of $\mathcal{F}_r$ and sides of $\mathcal{S}_r$ for an infinitesimally small r in $O(m \log m + M)$ time using a line segment intersection algorithm [3, 4, 6]. We claim that the smallest r that causes a violation of condition 7 can be determined by checking each straight side of $\mathcal{S}$ separately in $O(1)$ time, assuming that conditions 1 and 2 are not violated. Consider two values r, r' with $r < r'$ and assume that for both values conditions 1 and 2 hold. Then we have that if a straight side is in $\mathcal{S}_{r'}$, then the corresponding side also occurs in $\mathcal{S}_r$. In other words, when increasing r, sides may disappear from $\mathcal{S}_r$, but they cannot appear. The first violation of condition 7 happens exactly at the lowest value of r where a straight side does not occur anymore. Since the test at which value of r a side disappears due to the two adjacent sides can easily be performed in $O(1)$ time, condition 7 can be handled in $O(m \log m + M)$ time.

For conditions 4 and 6, we observe by Lemma 1 that they cannot be violated for any value of r as long as conditions 1, 2, 3, 5, and 7 are not violated. Hence we need not consider them.

Finally, for condition 5, we observe that by the assumption $r/w \geq 1$ and the assumption that condition 7 is not violated, the intersection of two edges (not incident to the same vertex) cannot cause such a violation. The only possibility of a violation of condition 5 is due to at least three edges incident to the same vertex which are together (in the meaning of Section 3). For a vertex and its incident edges, the value of r does not influence whether condition 5 is violated or not, since r/w is fixed. Hence, we compute the union of all vertices (disks) and edges (rectangles) for a sufficiently small r; this is the complement of $\mathcal{F}_r$. Then we remove the disks that are the vertices, and compute the Voronoi diagram of the sides (boundary parts) of this union minus the disks. The largest enclosed disk has its center on a Voronoi vertex, and hence we can decide if condition 5 is violated. The test takes $O(M \log M)$ time.

Theorem 3 *For a given straight-line drawing of a graph with n vertices and m edges, vertex radius r and edge width w, if $r/w \geq 1$ is fixed, then we can compute the smallest value of r that causes one of the seven conditions for a good bold drawing to be violated in $O^*(M + m \cdot n^{1/3} + n^{4/3})$ or $O^*(M + n^2)$ time, where M is the complexity of the arrangement of the drawing if $r = w = 0$.*

5 Discussion

We have taken a geometric look at drawings of graphs, and gave a list of conditions that may be used to define a good drawing of a graph with disks as vertices and rectangles as edges. Perhaps the most interesting result we proved is for what ratios of disk radius and rectangle width we cannot have drawings with vertices of arbitrarily high degree. We also gave algorithmic results on computing the largest radius and width while not violating the conditions. In the full paper we also study drawings that satisfy conditions 1–6, as condition 7 may be considered too strong.

Several questions arise from our research. The seven conditions do not capture ambiguity perfectly. Does a set of conditions exist that precisely captures ambiguity? Secondly, if we assume that r/w is bounded by some constant, does any graph (even K_n) have a drawing that satisfies the seven conditions? What if we use the conditions 1–6 but not 7? Finally, what are good bold drawing conditions if vertices are shown as disks filled white, or if they are shown as grey disks?

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