An Inequality on the Edge Lengths of Triangular Meshes*

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Abstract

We give a short proof of the following geometric inequality: for any two triangular meshes A and B of the same polygon C, if the number of vertices in A is at most the number of vertices in B, then the maximum length of an edge in A is at least the minimum distance between two vertices in B. Here the vertices in each triangular mesh include the vertices of the polygon and possibly additional Steiner points. The polygon must not be self-intersecting but may be non-convex and may even have holes. This inequality is useful for many purposes, especially in proving performance guarantees of mesh generation algorithms. For example, a weaker corollary of the inequality confirms a conjecture of Aurenhammer et al. [Theoretical Computer Science 289 (2002), 879–895] concerning triangular meshes of convex polygons, and improves the approximation ratios of their mesh generation algorithm for minimizing the maximum edge length and the maximum triangle perimeter of a triangular mesh.

1 Introduction

We prove the following theorem:

Theorem 1 *Let A and B be two triangular meshes of the same polygon C (which must not be self-intersecting but may be non-convex and may even have holes). If the number of vertices in A is at most the number of vertices in B, then the maximum length of an edge in A is at least the minimum distance between two vertices in B.*

Triangulation and mesh generation are fundamental problems in computational geometry [2]. Most previous algorithms for mesh generation focus on quality measures that either maximize the minimum angle or minimize the maximum angle of a triangular mesh because, in the predominant application to finite element analysis, triangular meshes should have neither too small nor too large angles. The triangular meshes generated by such algorithms are guaranteed to have bounded maximum-to-minimum angle ratios, but not bounded maximum-to-minimum edge length ratios.

Recognizing the importance of length-uniform triangular meshes in certain applications, Aurenhammer et al. [1] studied the problem of approximating length-uniform triangular meshes under the following three optimality criteria: (i) minimizing the maximum-to-minimum edge length ratio, (ii) minimizing the maximum edge length, and (iii) minimizing the maximum triangle perimeter. They proposed an efficient algorithm that, given a convex polygon P and a positive integer n, triangulates P using n Steiner points. The algorithm first applies a dispersion heuristic to select the Steiner points, next constructs the Delaunay triangulation of the polygon using the selected Steiner points, and finally modifies the Steiner triangulation into a triangular mesh that, with some reasonable assumptions on the input, achieves a constant approximation ratio for each of the three criteria.

For a convex polygon P and a positive integer n, define

$$d_{\text{long}} = \min_{T} \max_{e \in E(T)} \text{length}(e),$$

where T ranges over all Steiner triangulations of P with n Steiner points, and $E(T)$ is the set of edges in the triangulation T. Also define

$$d^* = \max_{S} \min_{u,v \in S \cup V(P)} \text{distance}(u, v),$$

where S ranges over all sets of n Steiner points in P, and $V(P)$ is the set of vertices of the polygon P.

The approximation ratios of Aurenhammer et al.'s algorithm (for minimizing the maximum edge length and the maximum triangle perimeter) crucially depend on the ratio of the two numbers d_{long} and d^*. By a simple area argument, Aurenhammer et al. were able to prove the inequality

$$d_{\text{long}} \geq \frac{\sqrt{3}}{2} d^*,$$

and they posed the following conjecture:

Conjecture 1 (Aurenhammer et al., 2002 [1]). $d_{\text{long}} \geq d^*$.

Let C be the convex polygon P. Let A be a Steiner triangulation of P with n Steiner points such that the maximum edge length is d_{long}. Let S be a set of n Steiner points in P that realizes the minimum pairwise distance d^* among the point set $S \cup V(P)$, and let B be any triangulation of the point set $S \cup V(P)$ such

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that the minimum edge length is d^*. Then Theorem 1 confirms Conjecture 1. As a consequence, the approximation ratios of Aurenhammer et al.'s mesh generation algorithm are immediately improved from $4\sqrt{3}$ to 6 for minimizing the maximum edge length, and from $6\sqrt{3}$ to 9 for minimizing the maximum triangle perimeter.

A natural question is whether the following two statements are also true:

1. For any two triangular meshes A and B of the same polygon C, if the number of vertices in A is at most the number of vertices in B, then the maximum edge length of A is at least the minimum edge length of B.

2. For any two triangular meshes A and B of the same polygon C, if the number of vertices in A is at most the number of vertices in B, then the maximum triangle perimeter of A is at least the minimum triangle perimeter of B.

Both statements turn out to be false. We give counter-examples in Figure 1 and Figure 2.

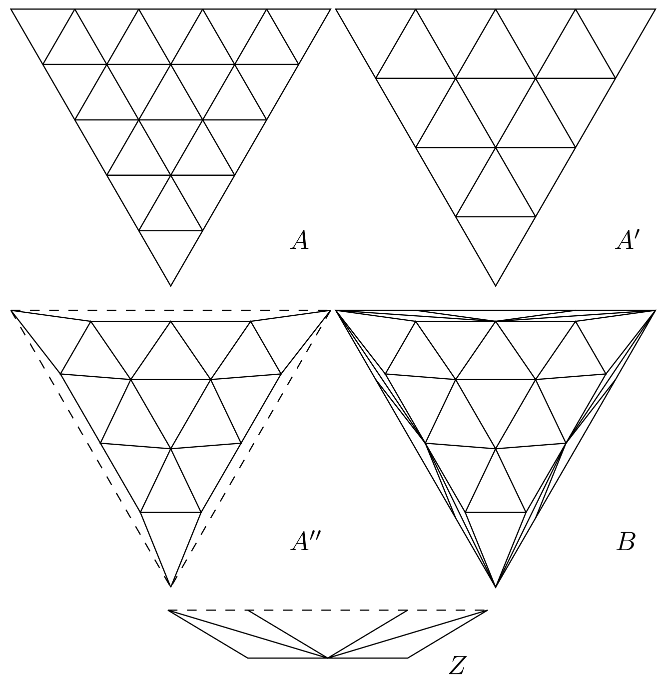

Figure 1: Two triangular meshes A and B of the same unit equilateral triangle C such that A and B have the same number of vertices but every edge of B is longer than every edge of A. A has $(5+1)(5+2)/2 = 21$ vertices and uniform edge length $1/5$. A' has $(4+1)(4+2)/2 = 15$ vertices and uniform edge length $1/4$. Move the Steiner points on the boundary slightly to change A' into A'' surrounded by three empty trapezoids. Add two more Steiner points to each side of the unit equilateral triangle, and triangulate each trapezoid Z. A'' and the three rotated copies of Z together form B, which has $15 + 2 \cdot 3 = 21$ vertices and minimum edge length close to $1/4$. This construction can be generalized: for each $k \geq 5$, there is a mesh A with $(k+1)(k+2)/2$ vertices and uniform edge length $1/k$, and there is a mesh B with $k(k+1)/2 + 3(k-3) = (k+1)(k+2)/2 + 2(k-5)$ vertices and minimum edge length close to $1/(k-1)$.

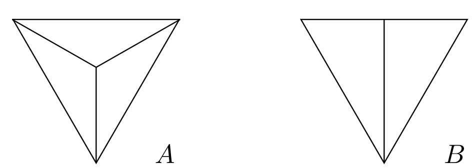

Figure 2: Two triangular meshes A and B of the same unit equilateral triangle C. Each mesh has four vertices: three vertices of the triangle and one Steiner point. The maximum triangle perimeter of A is $1 + 2\frac{\sqrt{3}}{3} = 2.1547\ldots$. The minimum triangle perimeter of B is $1 + \frac{1}{2} + \frac{\sqrt{3}}{2} = 2.3660\ldots$.

2 Proof of Theorem 1

We first introduce some preliminaries. A *d-simplex* is the convex hull of $d+1$ affinely independent vertices (that is, $d+1$ points in general position) in some Euclidean space of dimension d or higher. For example, in the plane, a 0-simplex is a point, a 1-simplex is a line segment, and a 2-simplex is a triangle. A simplex σ is a *face* of another simplex τ if the vertices of σ are a subset of the vertices of τ. A *simplicial complex* is a set K of simplices such that (i) any face of a simplex in K is also a simplex in K, and (ii) the intersection of any two simplices σ and τ in K is a face of both σ and τ.

For a simplicial complex K in the plane, denote by $\alpha_r(K)$ the number of r-simplices in K, $0 \leq r \leq 2$. Define the *Euler characteristic* of K as $\chi(K) = \alpha_0(K) - \alpha_1(K) + \alpha_2(K)$. For example, in Figure 2, we have $\alpha_0(A) = 4$, $\alpha_1(A) = 6$, $\alpha_2(A) = 3$, $\alpha_0(B) = 4$, $\alpha_1(B) = 5$, $\alpha_2(B) = 2$, and $\chi(A) = \chi(B) = 1$. For a 1-simplex σ, denote by $|\sigma|$ the length of σ, and denote by $\varepsilon(\sigma, K)$ the number of 2-simplices in K having σ as a face. Then $\varepsilon(\sigma, K) = 0$, 1, or 2. Define the *area* of K as the total area of the 2-simplices in K. Define the *perimeter* of K as $\sum_\sigma (2 - \varepsilon(\sigma, K))|\sigma|$, where σ ranges over all 1-simplices in K.

A triangular mesh of a polygon can be viewed as a simplicial complex in the plane: the 2-simplices are the triangles, the 1-simplices are the edges, and the 0-simplices are the polygon vertices and the Steiner points. The Euler characteristic of the triangular mesh is exactly one minus the number of holes in the polygon. The area and the perimeter of the triangular mesh (as a simplicial complex) are respectively the same as the area and the perimeter (in the normal sense) of the polygon.

We will prove the contrapositive of Theorem 1. Denote by $\ell_{\max}(T)$ and $\delta_{\min}(T)$, respectively, the maximum length of an edge and the minimum distance between two vertices in a triangular mesh T. Let A and B be two triangular meshes of the same polygon C. Suppose that $\delta_{\min}(B) > \ell_{\max}(A)$. We will show that $\alpha_0(B) < \alpha_0(A)$.

Our proof will use the following lemma by Folkman and Graham [3], which is reminiscent of Pick's theorem on the area of a simple polygon with vertices of integer coordinates:

Lemma 2 (Folkman and Graham, 1969 [3]). *Let K be a simplicial complex in the plane. Suppose that the distance between any two 0-simplices in K is at least 1. Then the total number of 0-simplices in K is at most $\frac{2}{\sqrt{3}}\operatorname{area}(K) + \frac{1}{2}\operatorname{peri}(K) + \chi(K)$.*

We first bound the area of A. Each triangle in A has edge length at most $\ell_{\max}(A)$. For any triangle of edge length ℓ, we can transform it into an equilateral triangle of edge length exactly ℓ as follows. First move any vertex of the triangle perpendicularly away from the opposite edge, until one of the two edges incident to the vertex has length exactly ℓ, next extend the other incident edge until its length is also ℓ, and finally extend the opposite edge also to length ℓ. Note that the area of the triangle does not decrease during this transformation. Since an equilateral triangle of edge length ℓ has an area exactly $\frac{\sqrt{3}}{4}\ell^2$, it follows that each triangle in A has an area at most $\frac{\sqrt{3}}{4}\ell^2_{\max}(A)$. Thus the area of A is at most $\alpha_2(A) \cdot \frac{\sqrt{3}}{4}\ell^2_{\max}(A)$.

We next bound the perimeter of A. Denote by $\beta_0(T)$ the number of vertices in a triangular mesh T that are on the boundary of the underlying polygon, including the polygon vertices and possibly additional Steiner vertices on the boundary. For example, in Figure 2, we have $\beta_0(A) = 3$ and $\beta_0(B) = 4$. Since A has exactly $\beta_0(A)$ edges on the boundary of C, the perimeter of A is at most $\beta_0(A) \cdot \ell_{\max}(A)$.

We now derive an equality that links the four parameters $\alpha_2(A)$, $\alpha_0(A)$, $\beta_0(A)$, and $\chi(A)$. Note that each boundary edge of a triangular mesh is incident to one triangle, and that each internal edge of a triangular mesh is incident to two triangles; on the other hand, each triangle has three edges. Thus by double-counting we have

$$
\begin{aligned}
1 \cdot \beta_0(A) + 2 \cdot (\alpha_1(A) - \beta_0(A)) &= 3 \cdot \alpha_2(A) \\
\implies \alpha_1(A) &= \frac{3\alpha_2(A) + \beta_0(A)}{2}.
\end{aligned}
$$

Recall that $\chi(A) = \alpha_0(A) - \alpha_1(A) + \alpha_2(A)$. Thus

$$
\begin{aligned}
\chi(A) &= \alpha_0(A) - \frac{3\alpha_2(A) + \beta_0(A)}{2} + \alpha_2(A) \\
&\implies \alpha_2(A) = 2\alpha_0(A) - \beta_0(A) - 2\chi(A).
\end{aligned}
$$

Finally, to complete the proof, we have

$$
\begin{aligned}
&\alpha_0(B) \\
&\le \frac{2}{\sqrt{3}} \cdot \frac{\operatorname{area}(B)}{\delta^2_{\min}(B)} + \frac{1}{2} \cdot \frac{\operatorname{peri}(B)}{\delta_{\min}(B)} + \chi(B) \quad \text{(by Lemma 2)} \\
&= \frac{2}{\sqrt{3}} \cdot \frac{\operatorname{area}(A)}{\delta^2_{\min}(B)} + \frac{1}{2} \cdot \frac{\operatorname{peri}(A)}{\delta_{\min}(B)} + \chi(A) \\
&< \frac{2}{\sqrt{3}} \cdot \frac{\operatorname{area}(A)}{\ell^2_{\max}(A)} + \frac{1}{2} \cdot \frac{\operatorname{peri}(A)}{\ell_{\max}(A)} + \chi(A) \\
&\le \frac{2}{\sqrt{3}} \cdot \frac{\alpha_2(A) \cdot \frac{\sqrt{3}}{4}\ell^2_{\max}(A)}{\ell^2_{\max}(A)} + \frac{1}{2} \cdot \frac{\beta_0(A) \cdot \ell_{\max}(A)}{\ell_{\max}(A)} + \chi(A) \\
&= \frac{1}{2}\alpha_2(A) + \frac{1}{2}\beta_0(A) + \chi(A) \\
&= \frac{1}{2}\big(2\alpha_0(A) - \beta_0(A) - 2\chi(A)\big) + \frac{1}{2}\beta_0(A) + \chi(A) \\
&= \alpha_0(A),
\end{aligned}
$$

as required.

References

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[2] M. Bern. Triangulations and mesh generation. In *Handbook of Discrete and Computational Geometry*, 2nd edition, CRC Press, pages 563–582, 2004.

[3] J.H. Folkman and R.L. Graham. A packing inequality for compact convex subsets of the plane. *Canadian Mathematical Bulletin*, 12:745–752, 1969.