Wireless Localization with Vertex Guards is NP-hard

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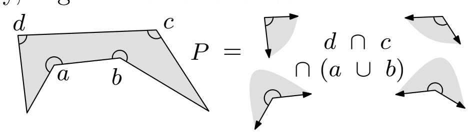
Abstract

We consider a special class of art gallery problems inspired by wireless localization. Given a simple polygon P, place and orient guards each of which broadcasts a unique key within a fixed angular range. In contrast to the classical art gallery setting, broadcasts are not blocked by the boundary of P. At any point in the plane one must be able to tell whether or not one is located inside P only by looking at the set of keys received. In other words, the interior of the polygon must be described by a monotone Boolean formula composed from the keys. We prove NP-hardness of several variants of the problem, in particular, for the vertex guard setting where guards must be located on vertices of P.

1 Introduction

We consider a new class of art gallery problems, introduced by Eppstein et al. [4]. They modify the concept of visibility by not considering edges of the polygon/gallery as opaque. This changes the problem drastically because it breaks up a certain locality where the polygon shape dictates possible placements of guards. An ingredient of hardness proofs for the classical setting is a small pocket of the polygon that can be guarded from a nearby point only because the polygon edges shield it away from the rest of the world. This argument breaks down if the edges do not block visibility.

The motivation for this model stems from communication in wireless networks where the signals are not blocked by walls, either. For illustration, suppose you run a café (modeled, say, as a simple polygon region P) and you want to provide wireless Internet access to your customers. But you do not want the whole neighborhood to use your infrastructure. Instead, Internet access should be limited to those people who are located within the café. To achieve this, you can install a certain number of devices, let us call them guards, each of which broadcasts a unique (secret) key in an arbitrary but fixed angular range. The goal is to place guards and adjust their angles in such a way that everybody who is inside the café can prove this fact just by naming the keys received and nobody who is outside the café can provide such a proof. Formally this means that P can be described by a monotone Boolean formula over the keys, that is, a formula using the operators AND and OR only, negation is not allowed.



Several different models for guard placement have been studied. Most restricted is a *natural guarding*, where every guard must be placed at a vertex of P and both its rays must be aligned with one of the incident edges. More general is a *vertex guarding*, where guards must be placed at vertices of P but rays may be chosen arbitrarily. Even less restricted is an *internal guarding*, in which guards can be placed anywhere inside P with no restriction on their rays. Finally, in a *general guarding* guards can be placed and oriented arbitrarily.

There are some results [4, 3, 2] concerning the minimum number of guards needed for a polygon on n vertices, but a tight bound $(n-2)$ is known for the natural setting only. On the negative side, we have shown recently [1] that deciding whether a collection of polygons (or a polygon with holes) can be guarded with k natural guards is NP-complete. In this paper, we prove that this problem is hard even for a single polygon, using a completely different reduction. Another benefit of the new reduction is that we can extend it to more general types of guards, such as vertex guards and internal guards.

2 Notation and Definitions

A *guard* g is a closed subset of the plane, whose boundary ∂g is described by a vertex v_g and two rays emanating from v_g. The ray that has the interior of the guard to its right is called the *left ray* ℓ_g, the other one is called the *right ray* r_g. The *angle* of a guard is the interior angle formed by its bounding rays. A *guarding* $\mathcal{G}$ of a simple polygon P is a set of guards such that there is a formula composed of this set and the operators union and intersection that defines P. A guard that is placed at a vertex of P is a *vertex guard*. A vertex guard is *natural* if it covers exactly the interior angle of its vertex. A guard placed anywhere on the line given by an edge of P and broadcasting within an angle of π to the inner side of the edge is called a *natural edge guard*. A *natural guarding* is a guarding consisting of natural vertex and natural edge guards only.

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A guard g *covers* an edge e of P (*completely*) if $e \subseteq \partial g$ and their orientations (inner sides) match. A guard that covers exactly k edges is a *k-guard*. A guard g *covers* an edge e of P *partly* if their orientations match and $e \cap \partial g$ is a proper sub-segment of e that is not just a single point. If there are no collinear edges, a guard can cover at most two edges; then a natural vertex guard is a 2-guard and a natural edge guard is a 1-guard. The line through an edge e of P is denoted by $\overline{e}$. The notion of guardings extends to *polygonal halfplanes*, that is, regions bounded by a *simple bi-infinite polygonal chain* (a polygonal chain that starts and ends with a ray).

Observation 1 [4] *For any guarding $\mathcal{G}$ of P and for any two points $p \in P$ and $q \notin P$ there is a guard $g \in \mathcal{G}$ which* distinguishes *p and q, that is, $p \in g$ and $q \notin g$.*

Observation 2 [2] *In any guarding $\mathcal{G}$ of a polygon P, every edge of P must be covered by at least one guard or it must be covered partly by at least two guards.*

3 Natural Wireless Localization is Hard

Theorem 1 *Given a simple polygons P and an integer k, it is an NP-complete problem to decide whether there exists a natural guarding for P using k guards.*

Given a simple polygon P and a set $\mathcal{G}$ of guards, we can decide in polynomial time if $\mathcal{G}$ is a guarding of P. (Consider the line arrangement induced by the edges of P and the rays of all guards in $\mathcal{G}$. Check for every pair (C, D) of cells of this arrangement with $C \subset P$ and $D \cap \text{int}(P) = \emptyset$ whether there is a guard $g \in \mathcal{G}$ that distinguishes them.) Therefore the problem is in NP.

To show NP-hardness we reduce from Monotone-SAT [5]. Let F be a monotone CNF formula with clauses $C_1, \ldots, C_m$ over variables $x_1, \ldots, x_n$, and denote $\deg(x_i) := |\{C_j : x_i \in C_j \text{ or } \overline{x_i} \in C_j\}|$. A clause is positive (negative) if all its literals are positive (negative).

The basic picture of the reduction is the following. We define different gadgets, which are bi-infinite polygonal chains. In the end we connect these gadgets to form a simple polygon. The variable gadget for a variable x_i is a merlon-like chain of length $4\deg(x_i) + 3$, which can be guarded optimally in essentially two ways, thus encoding the truth value of x_i. For every clause C_j there is a clause gadget of length 4. Any clause gadget can be guarded with 2 guards only if it is intersected by another guard ray. Depending on how a variable gadget is guarded, there are such guard rays, either to the positive or to the negative clauses the variable appears in. Finally, we put everything together to a simple polygon (Figure 3) using two intermediate chains.

Clause gadget. For every C_j we define a clause gadget R_j, which is a chain with 4 edges (Figure 1). Depending on whether C_j is positive or negative, R_j is of the first form or a vertical reflection of it. Such a chain cannot be guarded with two natural vertex guards. But it can be guarded with two guards if there is a ray of a third guard g intersecting it in the right way: $R_j = v_1 \cup (v_3 \cap g)$ or $R_j = (v_1 \cap g) \cup v_3$, respectively. Note that the "right" orientation of these additional rays is opposite for positive and negative clause gadgets.

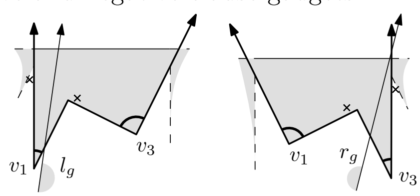

Figure 1: A positive and a negative clause gadget. The two crosses can only be distinguished with help of another correctly oriented ray crossing the gadget.

Variable gadget. For every variable x_i define a variable gadget Q_i (Figure 2), as a chain with edges $(e_1, \ldots, e_{k_i})$, where $k_i = 4\deg(x_i)+3$. There is a "spike" for every clause x_i appears in, first the positive clauses then the negative ones. If the clause is positive or negative, then the line through e_k, $k \equiv 3 \mod 4$ or $k \equiv 5 \mod 4$, respectively, intersects the clause gadget. Note that the orientation of these rays matches the needs of the corresponding clause gadgets. (This is where we use that clauses are monotone.)

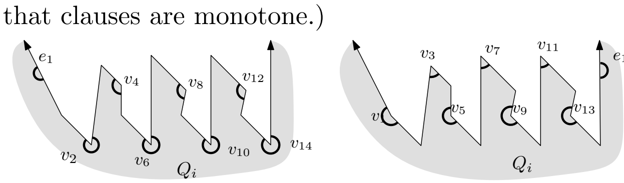

Figure 2: A positive and a negative guarding of Q_i. $Q_i = e_1 \cup (v_2 \cap (v_4 \cup v_6) \cap (v_8 \cup v_{10}) \cap (v_{12} \cup v_{14}))$, or $Q_i = v_1 \cup (v_3 \cap v_5) \cup \ldots \cup (v_{11} \cap v_{13}) \cup e_{15}$.

Connecting the gadgets. We define gadgets I_1 and I_2 which are simply used to connect everything (Figure 3). $P(F) = I_1 \cap ((R_1 \cup \ldots \cup R_m) \cup (I_2 \cap Q_1 \cap \ldots \cap Q_m))$. $P(F)$ has $4m + 10 + \sum_{i=1}^{n}(4\deg(x_i) + 3)$ edges in total.

Lemma 2 *If F is satisfiable, $P(F)$ can be guarded with $2m + 5 + \sum_{i=1}^{n}(2\deg(x_i) + 2)$ guards.*

Proof. Consider a satisfying assignment. Depending on the truth value of x_i we guard Q_i either positively or negatively with $2(\deg x_i + 1)$ guards (Figure 2). Consider a clause gadget R_j for a positive clause $C_j = \{x_{j_1}, x_{j_2}, x_{j_3}\}$. At least one of the variables $x_{j_1}, x_{j_2}, x_{j_3}$ is set to true. Thus there is a ray of a guard g from the corresponding variable gadget passing through the clause gadget with correct orientation. Therefore, R_j

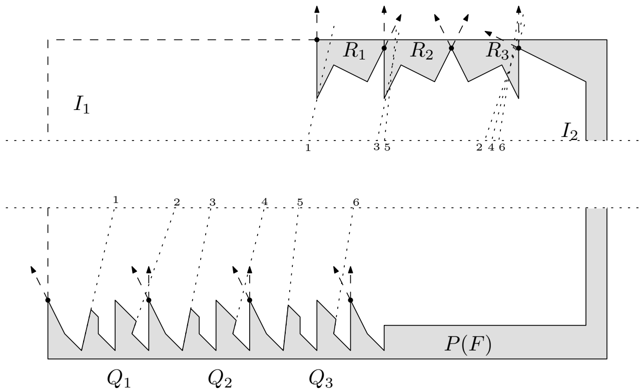

Figure 3: The polygon $P(F)$ for the formula $F = C_1 \wedge C_2 \wedge C_3 = x_1 \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$.

can be guarded using two natural vertex guards and g (Figure 1). Similarly, we can guard a negative clause gadget. Five more guards are needed for I_1 and I_2. □

Lemma 3 *If $P(F)$ can be guarded with $2m + 5 + \sum_{i=1}^{n}(2\deg(x_i) + 2)$ natural guards, then F is satisfiable.*

Proof. Let $\mathcal{G}$ be a guarding of $P(F)$ consisting of $2m + 5 + \sum_{i=1}^{n}(2\deg(x_i) + 2)$ guards. A guard *belongs* to a variable gadget if it is an edge guard on one of its edges or a natural vertex guard on one of its vertices or if it is the natural vertex guard at the intersection with the next chain to the left.

By Observation 2 every edge of the variable gadget has to be covered somehow. Except for the last edge only guards that belong to the gadget can do so. Since a guard can cover at most two edges, at least $2\deg(x_i)+1$ guards belong to the gadget. There is only one way to guard every edge except the last one with that many guards, namely using a natural vertex guard on every other vertex of the chain starting with the first vertex (Figure 4). But in this case there is no vertex guard on the last vertex and no edge guard on the last edge, hence there is no guard that can distinguish a point p near to the second edge of the next chain inside $P(F)$ and a point q near to the last edge of this chain outside $P(F)$. (There may be rays of guards that cross pq, but they cannot have the right orientation.) Therefore, there can be no such guarding and at least $2\deg(x_i) + 2$ guards belong to the gadget.

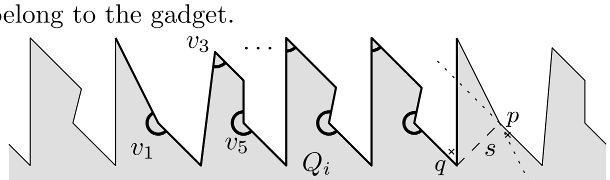

Figure 4: If only $2\deg(x_i)+1$ guards belong to Q_i they have to be exactly the ones shown here. But then, neither these guards nor guards belonging to other gadgets can distinguish p and q.

Intuitively, there is some freedom in how to guard a vertex gadget with $2\deg(x_i)+2$ guards because we have "half a guard" in excess. We can start with natural vertex guards on every other vertex and put a natural edge guard on the last edge (Figure 2 right) or we can start with an edge guard right away and then continue with natural vertex guards on every other vertex (Figure 2 left). Or we can do a combination of both, starting the first way and at some place put a natural vertex guard and continue in the second way. All possible guardings have one thing in common. Looking from left to right, we can change exactly once, from the first pattern to the second. As soon as we are in the second pattern, we cannot change (back) to the first without "paying" an additional guard. If there is a change to the second pattern within the positive spikes (such that at least one positive ray is emitted towards the corresponding clause gadget), the gadget is guarded *positively*; otherwise, the gadget is guarded *negatively*.

A guard belonging to a variable gadget can only cover edges of the variable gadgets. (An exception is the leftmost edge of $P(F)$, which might be covered by a natural vertex guard belonging to Q_1. But by considering a pair of points as shown in Figure 4, but now on the leftmost spike, we can argue that there must be a second guard covering this leftmost edge.) Thus the remaining $4m + 10$ edges have to be covered by the remaining $2m+5$ guards. There is only one possible way to achieve this: put a natural vertex guard on every other vertex.

A clause gadget can be guarded with two natural vertex guards iff there is another correctly oriented guard ray crossing it as depicted in Figure 1. The only rays that might do that are those emanating from guards covering the corresponding edge in a variable gadget of a variable that appears in the clause. At least one of these rays must be present, which means that the corresponding variable gadget must be guarded negatively or positively for a negative or positive clause, respectively. Therefore, we obtain a satisfying assignment as follows: If the gadget of a variable is guarded positively, we set the variable to true, if it is guarded negatively, we set it to false. □

4 A more General Setting

If guards can be located anywhere in the plane, in particular, on the intersections of two lines of the line arrangement outside the polygon, the usual arguments break down. But the situation improves if we forbid guards outside P. We call a guard whose vertex is inside P or on the boundary of P an *internal* guard.

The Internal Wireless Localization Problem Given a simple polygon P and a integer k, is there a guarding for P using k internal guards?

Theorem 4 *The Internal Wireless Localization Problem is NP-complete.*

Membership in NP follows in the same way as for the natural setting. To prove the NP-completeness we use a similar reduction as in the natural setting, but we have to change it a little bit. Intuitively, the problem is that for every variable gadget Q_i there is one guard that covers one edge only and its other ray is not "used". Now that we allow general guards, this unused ray is free to point to a clause gadget. In this way, clause gadgets could be guarded with 2 natural vertex guards even though none of its corresponding variable gadgets is guarded in the right way. We overcome this problem by introducing n additional *special gadgets* to bind these free rays.

Special gadget. We define n special gadgets, which are chains with 6 edges. A special gadget looks like a positive clause gadget rotated by $\pi/2$ in clockwise direction and with small spike added at the top. We include the special gadgets to the right. We define the variable gadgets $Q_1, \ldots, Q_n$ and the clause gadgets $R_1, \ldots, R_m$ essentially as in the natural setting. In the variable gadgets we add one additional spike at the beginning, so Q_i now consists of $4\deg(x_i) + 7$ edges. See Figure 5.

Observation 3 *The only 2-guards in a guarding of $P(F)$ are natural vertex guards.*

Lemma 5 *If F is satisfiable, $P(F)$ can be guarded with $2m + 3n + 6 + \sum_{i=1}^{n}(2\deg(x_i) + 4)$ guards.*

Proof. Depending on the truth values of x_i in a satisfying assignment we guard Q_i either positively or negatively with $2\deg x_i + 4$ guards similar to the natural setting (see Figure 2), but instead of just using natural edge guards we now use the "free" ray to help guarding one of the special gadgets, see Figure 5. Then, as in the natural setting, we can guard all the other gadgets using natural vertex guards only. □

Lemma 6 *If $P(F)$ can be guarded with $2m + 3n + 6 + \sum_{i=1}^{n}(2\deg(x_i) + 4)$ internal guards, then F is satisfiable.*

For the proof of Lemma 6 we refer to the appendix. The idea is the following. Assume we are given a guarding of $P(F)$ using $2m + 3n + 6 + \sum_{i=1}^{n}(2\deg(x_i) + 4)$ guards. Every guard has two rays. If we count all rays of guards and the edges of $P(F)$ that have to be covered, we find that there are n rays more than edges. In a first step we look at the special gadgets and see that they must use these n additional rays in some sense to be guarded properly. Therefore we have some control over the guarding. The majority of the rays is used to cover edges and the additional rays are bound to the special gadgets. Then we can proceed as in the natural setting and show that a variable gadget Q_i can essentially be guarded in two ways. Either there are rays of guards pointing to the positive clause gadgets of the positive clauses x_i appears in, or there are rays of guards pointing to the negative clause gadgets corresponding to x_i. Setting the truth values of the variables accordingly we find a satisfying assignment.

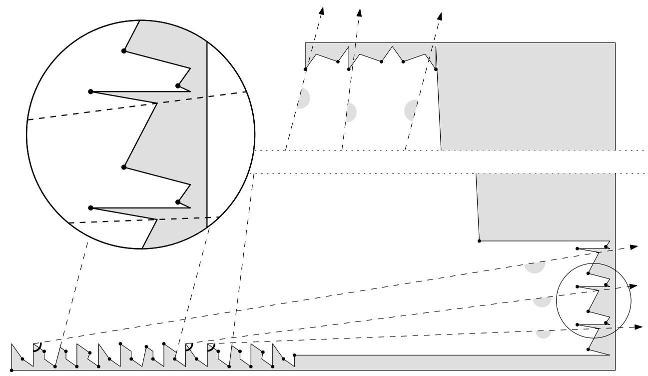

Figure 5: An optimal guarding of $P(F)$ corresponding to a satisfying assignment, the marked vertices are the positions of natural vertex guards.

The Wireless Localization Problem for Vertex Guards Given a simple polygons P and a integer k, is there a guarding for P using k vertex guards?

Corollary 7 *The wireless localization problem for vertex guards is NP-complete.*

Proof. The guarding given in Lemma 5 uses vertex guards only. Lemma 6 trivially remains true if we consider a guarding consisting of vertex guards only. □

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