Existence of zone diagrams in compact subsets of uniformly convex spaces

Eva Kopecká* Daniel Reem† Simeon Reich‡

Abstract

A zone diagram is a relatively new concept which has emerged in computational geometry and is related to Voronoi diagrams. Formally, it is a fixed point of a certain mapping, and neither its uniqueness nor its existence are obvious in advance. It has been studied by several authors, starting with T. Asano, J. Matoušek and T. Tokuyama, who considered the Euclidean plane with singleton sites, and proved the existence and uniqueness of zone diagrams there. We announce the existence of zone diagrams with respect to finitely many pairwise disjoint compact sites contained in a compact and convex subset of a uniformly convex normed space. The proof is based on the Schauder fixed point theorem, the Curtis-Schori theorem regarding the Hilbert cube, and on recent results concerning the characterization of Voronoi cells as a collection of line segments and their geometric stability with respect to small changes of the corresponding sites. Along the way we obtain interesting and apparently new properties of Voronoi cells.

1 Introduction

Background: A zone diagram is a relatively new concept related to geometry and fixed point theory. In order to understand it better, consider first the more familiar concept of a Voronoi diagram. In a Voronoi diagram we start with a set X, a distance function d, and a collection of subsets $(P_k)_{k\in K}$ in X (called the sites or the generators), and with each site P_k we associate the k-th Voronoi cell, that is, the set R_k of all $x \in X$ the distance of which to P_k is not greater than its distance to the union of the other sites P_j, $j \neq k$. On the other hand, in a zone diagram we associate with each site P_k the set R_k of all $x \in X$ the distance of which to P_k is not greater than its distance to the union of the other sets R_j, $j \neq k$. Figures 1 and 2 show the Voronoi and zone diagrams, respectively, corresponding to the same ten singleton sites in Euclidean plane. At first sight, it seems that the definition of a zone diagram

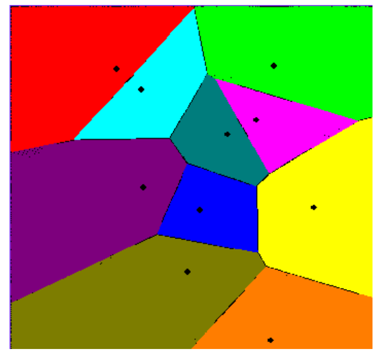

Figure 1: A Voronoi diagram of 10 point sites in a square in the Euclidean plane.

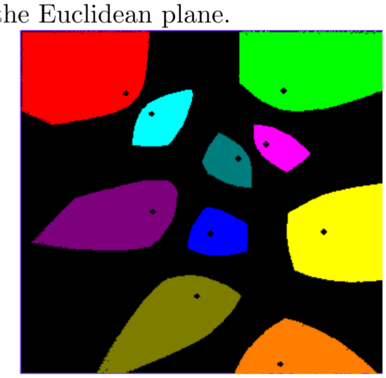

Figure 2: A zone diagram of the same 10 points as in Figure 1.

is circular, because the definition of each R_k depends on R_k itself via the definition of the other cells R_j, $j \neq k$. On second thought, we see that, in fact, a zone diagram is defined to be a fixed point of a certain mapping (called the Dom mapping), that is, a solution of a certain equation. While the Voronoi diagram is explicitly defined and, hence, its existence and uniqueness are obvious, neither the existence nor the uniqueness of a zone diagram are obvious in advance. As a result, in addition to the problem of finding algorithms for computing zone diagrams, we are faced with the more fundamental problem of establishing their existence (and uniqueness) in various settings, and with the problem of reaching a better understanding of this concept.

The concept of a zone diagram was first defined and studied by T. Asano, J. Matoušek and T. Tokuyama [3] (see also [2]), in the case where (X, d) was the Eu-

*Institute of Mathematics, Czech Academy of Sciences, Žitná 25, CZ-11567 Prague, Czech Republic, and Institut für Analysis, Johannes Kepler Universität, Altenbergerstrasse 69, A-4040 Linz, Austria, eva@bayou.uni-linz.ac.at

†Department of mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel, dream@tx.technion.ac.il

‡Department of mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel, sreich@tx.technion.ac.il

clidean plane, each site P_k was a single point, and all these (finitely many) points were different. They proved the existence and uniqueness of a zone diagram in this case, and also suggested a natural iterative algorithm for approximating it. Their proofs rely heavily on the above setting.

Later, the authors of [19] considered general sites in abstract spaces, called m-spaces, in which X is an arbitrary nonempty set and the "distance" function should only satisfy the condition $d(x,x) \leq d(x,y) \;\; \forall x,y \in X$ and can take any value in the interval $[-\infty, \infty]$. They introduced the concept of a double zone diagram, and using it and the Knaster-Tarski fixed point theorem, proved the existence of a zone diagram with respect to any two sites in X. They also showed that in general the zone diagram is not unique. In a recent work by K. Imai, A. Kawamura, J. Matoušek, Y. Muramatsu and T. Tokuyama [12], the existence and uniqueness of the zone diagram with respect to any number of general positively separated sites in the n-dimensional Euclidean space $\mathbb{R}^n$ were announced. The proof is based on results from [19] and on an elegant geometric argument specific to Euclidean spaces. Very recently some of these authors have generalized this result to finite dimensional normed spaces which are both strictly convex and smooth [14].

We note that zone diagrams are closely related to the concepts of trisector and k-gradation; see [1, 2, 4, 7, 13] for more details.

Our contribution: In this extended abstract we announce the existence of zone diagrams with respect to finitely many pairwise disjoint compact sites contained in a compact and convex subset of a (possibly infinite dimensional) uniformly convex normed space (full proofs can be found in [15]). The proof is based on the Schauder fixed point theorem, the Curtis-Schori theorem regarding the Hilbert cube, and on recent results concerning the characterization of Voronoi cells as a collection of line segments and their geometric stability with respect to small changes of the corresponding sites. Along the way we obtain interesting results regarding Voronoi cells in uniformly convex spaces. Although Voronoi diagrams have been the subject of extensive research during the last decades [5, 16], this research has been mainly focused on Euclidean finite dimensional spaces (in many cases just $\mathbb{R}^2$ or $\mathbb{R}^3$), and it seems that our results are new even for $\mathbb{R}^2$ with a non-Euclidean norm.

It may be of interest to compare our main existence result with the recent existence result described in [14]. On the one hand, our result is weaker than that result, since we only prove the existence of a zone diagram in a compact and convex set, while in [14] uniqueness is also proved and the setting is the whole space $\mathbb{R}^n$. In addition, it seems that some of the arguments in [14], although only formulated for finitely many sites, can be extended to infinitely many, positively separated closed sites. On the other hand, our result is stronger in the sense that we allow infinite dimensional spaces and we do not require the smoothness of the norm. As a matter of fact, the counterexamples mentioned in [14] show that uniqueness does not necessarily hold if the norm is not smooth. In any case, the strategies used for proving these two results are completely different: in [14] the authors use the existence of double zone diagrams (based on the Knaster-Tarski fixed point theorem) and several geometric arguments, and here we use the Schauder fixed point theorem, the Curtis-Schori theorem regarding the Hilbert cube, and several general results about Voronoi cells in uniformly convex normed spaces.

2 Definitions and Notation

We consider a compact and convex set $X \neq \emptyset$ in a uniformly convex normed space $(\widetilde{X}, |\cdot|)$, real or complex, finite or infinite dimensional. The induced metric is $d(x,y) = |x-y|$.

Definition 1 *Given two nonempty sets $P, A \subseteq X$, the dominance region* $\mathrm{dom}(P,A)$ *of P with respect to A is the set of all $x \in X$ the distance of which to P is not greater than its distance to A, that is,*

$$\mathrm{dom}(P,A) = \{x \in X : d(x,P) \leq d(x,A)\}.$$

Here $d(x,A) = \inf\{d(x,a) : a \in A\}$.

Definition 2 *Let K be a set of at least 2 elements (indices), possibly infinite. Given a tuple $(P_k)_{k \in K}$ of nonempty subsets $P_k \subseteq X$, called the generators or the sites, the Voronoi diagram induced by this tuple is the tuple $(R_k)_{k \in K}$ of nonempty subsets $R_k \subseteq X$, such that for all $k \in K$,*

$$\begin{aligned} R_k &= \mathrm{dom}(P_k, \bigcup_{j \neq k} P_j) \\ &= \{x \in X : d(x, P_k) \leq d(x, P_j) \;\forall j \neq k,\, j \in K\}. \end{aligned}$$

In other words, each R_k, called a Voronoi cell, is the set of all $x \in X$ the distance of which to P_k is not greater than its distance to the union of the other P_j, $j \neq k$.

Definition 3 *Let (X,d) be a metric space and let K be a set of at least 2 elements (indices), possibly infinite. Given a tuple $(P_k)_{k \in K}$ of nonempty subsets $P_k \subseteq X$, a zone diagram with respect to that tuple is a tuple $R = (R_k)_{k \in K}$ of nonempty subsets $R_k \subseteq X$ such that*

$$R_k = \mathrm{dom}(P_k, \bigcup_{j \neq k} R_j) \quad \forall k \in K.$$

In other words, if we define $X_k = \{C : P_k \subseteq C \subseteq X\}$, then a zone diagram is a fixed point of the mapping

$\mathrm{Dom}: \prod_{k\in K} X_k \to \prod_{k\in K} X_k$, *defined by*

$$\mathrm{Dom}(R) = (\mathrm{dom}(P_k, \bigcup_{j\neq k} R_j))_{k\in K}.$$

We now recall the definition of strictly and uniformly convex spaces.

Definition 4 *A normed space $(\widetilde{X}, |\cdot|)$ is said to be strictly convex if for all $x, y \in \widetilde{X}$ satisfying $|x| = |y| = 1$ and $x \neq y$, the inequality $|(x+y)/2| < 1$ holds. $(\widetilde{X}, |\cdot|)$ is said to be uniformly convex if for any $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that for all $x, y \in \widetilde{X}$, if $|x| = |y| = 1$ and $|x - y| \geq \epsilon$, then $|(x + y)/2| \leq 1 - \delta$.*

A uniformly convex space is always strictly convex, and if it is also finite dimensional, then the converse is true too. The n-dimensional Euclidean space $\mathbb{R}^n$, or more generally, inner product spaces, the sequence spaces ℓ_p, the Lebesgue spaces $L_p(\Omega)$, $p \in (1, \infty)$, are all examples of uniformly convex spaces. The plane $\mathbb{R}^2$ endowed with the max norm $|\cdot|_\infty$ is a typical example of a space which is not uniformly convex, since its unit sphere contains line segments. More information regarding uniformly convex spaces can be found, in, e.g., [6, 9].

We finish this section by recalling three definitions of a topological character.

Definition 5 *The Hilbert cube I^∞ is the set $I^\infty = \prod_{n=1}^{\infty}[0, 1/n]$ as a topological space the topology of which is induced by the ℓ_2 norm, or, equivalently, by the product topology.*

Definition 6 *A topological space X is said to be locally (path) connected if for any $x \in X$ and any open set U containing x, there exists a (path) connected open set $V \subseteq U$ such that $x \in V$.*

Definition 7 *Let (X, d) be a metric space. Given two nonempty sets $A_1, A_2 \subseteq X$, the Hausdorff distance between them is defined by*

$$D(A_1, A_2) = \max\{\sup_{a_1\in A_1} d(a_1, A_2), \sup_{a_2\in A_2} d(a_2, A_1)\}.$$

Recall that the Hausdorff distance is different from the usual distance between two sets which is defined by $d(A_1, A_2) = \inf\{d(a_1, a_2) : a_1 \in A_1, a_2 \in A_2\}$.

3 The main result and an outline of its proof

In this section we outline the proof of our main result, stated as follows:

Theorem 8 *There exists a zone diagram with respect to finitely many pairwise disjoint compact sites in any compact and convex subset of a uniformly convex normed space.*

The idea of the proof is to find a certain space Y homeomorphic to the Hilbert cube I^∞ ($Y \approx I^\infty$ for short) such that $\mathrm{Dom}(Y) \subseteq Y$ and Dom is continuous on Y. Now, if $h : I^\infty \to Y$ is a homeomorphism, then $f = h^{-1} \circ \mathrm{Dom} \circ h : I^\infty \to I^\infty$ is a continuous mapping which maps a compact and convex subset of ℓ_2 into itself, so the Schauder fixed point theorem [20] (see also [10, p. 119] and Theorem 9 below) ensures that f has a fixed point $q \in I^\infty$. By taking $R = h(q)$, we see that R is a fixed point of Dom, that is, R is a zone diagram. In order to apply this idea, one has to find the set Y, to prove that it is homeomorphic to I^∞, and to prove the continuity of Dom on Y. It has turned out that even in the case of singleton sites in a square in the Euclidean plane the proof is not obvious (the main difficulty is to prove the continuity of Dom), and, in fact, such a proof has never been published.

The above strategy was suggested by the first author, and was briefly mentioned in [3, p. 1188]. The space Y was taken to be $\prod_{k\in K} Y_k$, where K was finite, Y_k was $\{C : P_k \subseteq C \subseteq Q_k \text{ and } C \text{ is closed}\}$ and Q_k was the intersection of the k-th Voronoi cell with X (the square). Since each site P_k is taken to be a singleton, it follows that each Q_k is actually convex, so, in particular, it is a connected and locally connected compact metric space. Since, in addition, $P_k \neq Q_k$, it follows from the theorem of D. Curtis and R. Schori [8, Theorem 5.2] stated below (see also [11, p. 91]) that Y_k, as a metric space endowed with the Hausdorff metric, is homeomorphic to I^∞. The topology on Y is the product topology, induced by the uniform Hausdorff metric $\widetilde{D}((S_k)_{k\in K}, (S'_k)_{k\in K}) = \max\{D(S_k, S'_k) : k \in K\}$, so Y, as a finite product of spaces homeomorphic to I^∞, is also homeomorphic to I^∞.

Theorem 9 (Schauder) *Let X be a nonempty convex and compact subset of a normed space. If $f : X \to X$ is continuous, then it has a fixed point.*

Theorem 10 (Curtis-Schori) *Let X be a Peano continuum, that is, a connected and locally connected compact metric space, and let $P \subseteq X$, $P \neq X$ be closed and nonempty. Let $2^X_P = \{C : P \subseteq C \subseteq X, C \text{ is closed}\}$, endowed with the Hausdorff metric. Then $2^X_P \approx I^\infty$.*

In the general case, the application of the above strategy, and, in particular, the verification of the hypotheses of Theorem 10, are not a simple task, and they require several additional tools related to dominance regions, such as their characterization as unions of line segments, and their stability with respect to small perturbations of the relevant sets. These results have recently been established in [17, 18], and their proofs can be found there. Using these results, we first prove the existence of a zone diagram with respect to finite sites, and then, approximating compact sets by finite subsets of them

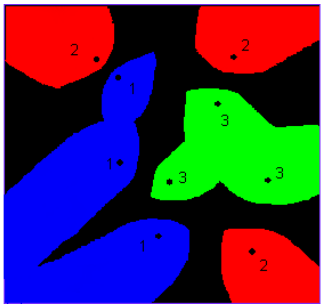

Figure 3: A zone diagram of 3 sites in a square in $(\mathbb{R}^2, \ell_p)$, $p = 3.14159$, each site consisting of 3 points.

and applying a continuity argument, we extend this existence result to any compact sites. Along the way we obtain several interesting properties of Voronoi cells in uniformly convex spaces: if $d(p, A) > 0$, then $\text{dom}(p, A)$ is path connected and locally path connected, and if, in addition, p is in the interior of the universe X, then $\text{dom}(p, A)$ is homeomorphic to a convex set.

4 Concluding remarks

In this extended abstract we have mainly focused on discussing the concept of zone diagrams and on the fundamental problem of proving their existence. It may be of interest to ask whether there exist methods for approximating them. It turns out that there is such a method, and it is based on a new and general algorithm for computing Voronoi diagrams [17, 18]. As a matter of fact, Figures 2 and 3 were obtained by using it. See [18] for more details.

Acknowledgements

We thank Akitoshi Kawamura, Jiří Matoušek, and Takeshi Tokuyama for providing a copy of their recent paper [14] before posting it on the arXiv, Orr Shalit for a useful suggestion, and one of the reviewers for several corrections. The first author was supported by Grant FWF-P19643-N18. The third author was partially supported by the Israel Science Foundation (Grant 647/07), the Fund for the Promotion of Research at the Technion (Grant 2001893), and by the Technion President's Research Fund (Grant 2007842).

References

[1] T. Asano and D. Kirkpatrick, *Distance trisector curves in regular convex distance metrics*, Proceedings of the 3rd International Symposium on Voronoi Diagrams in Science and Engineering (ISVD 2006), pp. 8–17.

[2] T. Asano, J. Matoušek, and T. Tokuyama, *The distance trisector curve*, Adv. Math. **212** (2007), 338–360, a preliminary version in STOC 2006, pp. 336-343.

[3] T. Asano, J. Matoušek, and T. Tokuyama, *Zone diagrams: Existence, uniqueness, and algorithmic challenge*, SIAM J. Comput. **37** (2007), no. 4, 1182–1198, a preliminary version in SODA 2007, pp. 756-765.

[4] T. Asano and T. Tokuyama, *Drawing equally-spaced curves between two points*, Abstracts of the 14th Annual Workshop on Computational Geometry with a Focus on Open Problems, 2004, pp. 24–25.

[5] F. Aurenhammer, *Voronoi diagrams - a survey of a fundamental geometric data structure*, ACM Computing Surveys, vol. 3, 1991, pp. 345–405.

[6] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.

[7] J. Chun, Y. Okada, and T. Tokuyama, *Distance trisector of segments and zone diagram of segments in a plane*, Proceedings of the 4th International Symposium on Voronoi Diagrams in Science and Engineering (ISVD 2007), pp. 66–73.

[8] D. W. Curtis and R. M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. **101** (1978), no. 1, 19–38.

[9] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker Inc., New York, 1984.

[10] J. Dugundji and A. Granas, *Fixed point theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[11] A. Illanes and S. B. Nadler Jr, *Hyperspaces. Fundamentals and recent advances*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 216, Marcel Dekker Inc., New York, 1999.

[12] K. Imai, A. Kawamura, J. Matoušek, Y. Muramatsu, and T. Tokuyama, *Distance k-sectors and zone diagrams*, EuroCG 2009, pp. 191–194.

[13] K. Imai, A. Kawamura, J. Matoušek, D. Reem, and T. Tokuyama, *Distance k-sectors exist*, Computational Geometry, accepted for publication, a preliminary version in SoCG 2010, pp. 210–215.

[14] A. Kawamura, J. Matoušek, and T. Tokuyama, *Zone diagrams in Euclidean spaces and in other normed spaces*, arXiv 0912.3016 (2009), a short version in SoCG 2010, pp. 216-221.

[15] E. Kopecká, D. Reem, and S. Reich, *Zone diagrams in compact subsets of uniformly convex normed spaces*, Israel Journal of Mathematics, accepted for publication, arXiv 1002.3583 (2010).

[16] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial tessellations: concepts and applications of Voronoi diagrams*, second ed., Wiley Series in Probability and Statistics, John Wiley & Sons Ltd., Chichester, 2000.

[17] D. Reem, *An algorithm for computing Voronoi diagrams of general generators in general spaces*, preprint, a short version in ISVD 2009, pp. 144-152.

[18] D. Reem, *Voronoi and zone diagrams*, Ph.D. thesis, The Technion, Haifa, 2010.

[19] D. Reem and S. Reich, *Zone and double zone diagrams in abstract spaces*, Colloquium Mathematicum **115** (2009), 129–145, arXiv 0708.2668 (2007).

[20] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171–180.