k-Sets and Continuous Motion in $\mathbb{R}^3$

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Abstract

We prove several new results concerning k-sets of point sets on the 2-sphere (equivalently, for *signed* point sets in the plane) and k-sets in 3-space. Specific results include spherical generalizations of (i) Lovász' lemma (regarding the number of spherical k-edges intersecting a given great circle) and of (ii) the crossing identity for k-edges due to Andrzejak et al. As a new ingredient compared to the planar case, the latter involves the winding number of k-facets around a given point in 3-space, as introduced by Lee and by Welzl, independently. As a corollary, we obtain a crossing identity for the number of pinched crossings (crossing pairs of triangles sharing one vertex) of k-facets in 3-space.

1 Introduction

Given a set P of n points in general position in $\mathbf{R}^d$ and an integer parameter k, a *k-set* of P is a subset $S \subseteq P$ of size $|S| = k$ that can be strictly separated from its complement $P \setminus S$ by an affine hyperplane. The *k-set problem* is: What is the maximum number of k-sets of an n-element point set in $\mathbf{R}^d$? This has been a basic open problem in discrete and computational geometry for more than thirty years[1], and despite intensive research, there still remain substantial gaps between the known upper and lower bounds, even in low dimensions.

In the plane $\mathbf{R}^2$, the currently best bound of $O(n\sqrt[3]{k})$ is due to Dey [7]. Like most papers dealing with k-sets, Dey does not work directly with k-sets but with the equivalent (up to constant factors) notion of *k-edges*, i.e., directed edges pq, spanned by points $p, q \in P$, such that there are exactly k points of P to the right of (the line through) the segment pq. A key ingredient in Dey's proof is to show that the number of crossings between k-edges is at most $O(n(k+1))$. Dey's analysis was further refined by Andrzejak et al. [4] who proved the following *crossing identity* for k-edges:

$$\mathrm{cr}_k(P) + \sum_{q\in P} \binom{\deg_k(q)}{2} = e_{<k}(P). \qquad (1)$$

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[1] Some of the key references are[14, 10, 11, 9, 17, 5, 23, 2, 8, 7, 4, 19, 20, 16]; for further references and background, we refer to the survey [21] or to [15, Chapter 11].

Here, cr_k is the number of crossings of k-edges cr_k, $\deg_k(q)$ is half of the number of k-edges emanating from q, and $e_{<k}$ is the total number of j-edges, $0 \le j < k$. This identity implies the desired bound on cr_k, since it is known [3, 18] that $e_{<k} \le nk$.

In dimension 3, one considers k-facets: triangles spanned by three points of P with precisely k points on a specified side. The best upper bound $O(nk^{3/2})$, due to Sharir et al. [19], is also proved by analyzing certain crossing configurations between k-facets, namely *pinched crossings* (two triangles that share a vertex and intersect in their relative interiors), but an exact identity like (1) is still missing.

Here, we prove such an identity, which we believe adds to our understanding of k-facets in $\mathbf{R}^3$. As a key technical step along the way, we first extend (1) to point sets on the 2-sphere. As a new ingredient compared to the planar case, our identities involve the winding number of k-facets around a given point in 3-space, as introduced by Lee [12] and by Welzl [22].

2 Basics and preliminaries

Points and vectors

In the rest of the text we will clearly differentiate between *point sets* with no distinguished origin where we are only interested in affine properties and *vector configurations* with a distinguished origin $\mathbf{0}$ where linear properties involving this origin come into play as well.

k-facets

Let P be a set of n points in $\mathbf{R}^d$ in *general position*, i.e., any $d+1$ or fewer points are affinely independent. Consider a $(d-1)$-dimensional simplex $\sigma := \sigma(p_1, \ldots, p_d) := \mathrm{conv}\{p_1, \ldots, p_d\}$ spanned by d points $p_1, \ldots, p_d \in P$. We will often identify the simplex σ with the set of points spanning it and write $\sigma = p_1 \ldots p_d$. The affine hull of σ is a hyperplane, which divides $\mathbf{R}^d$ into two open halfspaces. A *coorientation* of the simplex σ is declaring one of these two halfspaces positive and the other negative, denoted by σ^+ and σ^-, respectively. A cooriented simplex is called a *k-facet* if it contains exactly k points of P in its positive halfspace. In particular, the 0-facets of P are precisely the facets of its convex hull with coorientation given by the outer normal vectors. We denote the number of k-facets of P by $e_k(P)$, or simply e_k if P

is clear from the context. Note that $e_k = 0$ for $k < 0$ or $k > n-d$ and that $e_k = e_{n-d-k}$ for $k < \frac{n-d}{2}$ by simply reversing the coorientations. The number of $(\leq k)$-facets and $(< k)$-facets of P will be denoted by $e_{\leq k}(P)$ and $e_{<k}(P)$, respectively (i.e., $e_{\leq k}(P) := \sum_{i=0}^{k} e_i(P)$ and $e_{<k}(P) := \sum_{i=0}^{k-1} e_i(P)$) or only by $e_{\leq k}$ and $e_{<k}$ if the point set P is understood from the context.

Spherical k-facets

The d-dimensional unit sphere in $\mathbf{R}^{d+1}$ centered at origin $\mathbf{0}$ will be denoted by $\mathbf{S}^d$ (for example, $\mathbf{S}^1$ is the unit circle in the plane). Let V be a set of n vectors on $\mathbf{S}^d$ in *linearly general position* (i.e. any $d+1$ or fewer points linearly independent). Given a set of d vectors $v_1, \ldots, v_d \in V$, we will call the set $\sigma := \sigma(v_1, \ldots, v_d) := \mathbf{S}^d \cap \text{cone}\{v_1, \ldots, v_d\}$ a *spherical simplex* spanned by those vectors. There is a unique $(d-1)$-dimensional unit sphere S centered at $\mathbf{0}$ containing σ and it subdivides $\mathbf{S}^d$ into two open hemispheres. As we will deal with low-dimensional vector sets, this sphere will be called an equator of $\mathbf{S}^d$. A coorientation of σ is again declaring one of the hemispheres positive and the other negative, denoted by σ^+ and σ^-, respectively. A cooriented spherical simplex is called a *(linear) k-facet*, if it contains exactly k points of V in its positive hemisphere σ^+. A spherical k-facet $v_1 \ldots v_d$ defines a k-facet $v_1 \ldots v_d\mathbf{0}$ (with consistent coorientation) incident to $\mathbf{0}$ in the point set $V \cup \{\mathbf{0}\} \subseteq \mathbf{R}^{d+1}$ and vice versa.

Planar k-facets and linear k-facets on the two-dimensional sphere are called k-edges and an orientation of a k-edge by convention uniquely defines its coorientation by declaring the right halfplane to be positive and vice versa, a coorientation defines the orientation.

Crossings

For a set P of n points in $\mathbf{R}^d$ (or vectors in $\mathbf{S}^d$) and $q_1, \ldots, q_{d-1} \in P$, the $\deg_k(q_1 \ldots q_k)$ denotes half the number of k-facets of P containing $q_1, \ldots, q_{d-1}$. In another words, if one looks at a hypersurface [2] Σ_k defined by the k-facets of P, then $\deg_k(q_1 \ldots q_{d-1})$ is just the number of sheets of Σ_k passing through the simplex $q_1 \ldots q_{d-1}$. Two *k-facets a and b cross*, if their relative interiors intersect[3] (denoted by $\text{relint}(a) \cap \text{relint}(b) \neq \emptyset$) and a *line ℓ crosses a k-facets a*, if $\ell \cap \text{relint}(a) \neq \emptyset$.

Let P be a set of n points in $\mathbf{R}^2$ or vectors on $\mathbf{S}^2$. The number of pairs of crossings of k-edges in P will be denoted by $\text{cr}_k(P) := |\{\{pq, rs\} \mid pq \neq rs \text{ are crossing } k\text{-edges}\}|$.

[2]This surface is indeed well defined, as a consequence of the interleaving property of the k-facets.

[3]The k-facets are $(d-1)$-dimensional objects in $\mathbf{R}^d$, therefore have an empty interior. Thus we need to speak of their relative interiors instead.

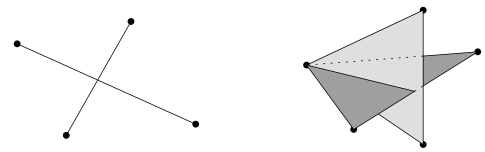

Figure 1: An edge crossing and a pinched crossing.

Let P be a set of n points in $\mathbf{R}^3$ or vectors on $\mathbf{S}^3$ and let opq, ors be two distinct k-facets of P. They form a *pinched crossing*, if $\text{relint}(opq) \cap \text{relint}(ors) \neq \emptyset$. The number of pairs of k-facets of P forming a pinched crossing will be denoted by $\text{pcr}_k(P)$.

Contractions

Let P be a set of points in $\mathbf{R}^d$ in general position and $S \subset P, |S| < d$. Denote P' a point set obtained by projecting $P \setminus S$ orthogonally onto $\text{aff}(S)^\perp$ (the affine orthogonal complement of the affine hull of S). Set the coordinate system such that the projection of S is $\mathbf{0}$. Now projecting P' centrally to a $(d-|S|)$-dimensional sphere yields a vector set on $\mathbf{S}^{d-|S|}$, which is the *contraction P/S*. Image of a point $p \in P$ or a set $Q \subset P$ under this mapping will be denoted p/S or Q/S respectively (assuming P is understood from the context).

Similarly, by taking the linear orthogonal complement of the linear span, we define contraction of a vector configuration.

Observe, that contraction U/S of a k-facet $U \cup S$ of P is again a k-facet in P/S and vice versa, if V is a k-facet of P/S then its preimage is a k-facet of P. Thus, k-facets of P/S are in one-to-one correspondence with k-facets of P containing S. For $q \in P$, the contraction $P/q := P/\{q\}$ is just the "angular view" of q onto $P \setminus \{q\}$. When $P \subset \mathbf{R}^3$ and $q \in P$, one can see, that k-facets in P which contain q and k-edges in P/q have the same "shape", i.e. two k-facets $U, V \ni q$ of P form a pinched crossing (centered at q) iff their contractions, k-edges U/q and V/q cross. If P is in convex position, the contraction P/q lies completely on one hemisphere of S^2 and thus, can be treated as a planar point set (for the purposes of k-edges and their crossings – as we can centrally project it on a plane close to q with the same normal vector as the aforementioned hemisphere).

f, g and h-vectors

Let V be a set of vectors in general position in $\mathbf{R}^d$. For $0 \leq k \leq n-d-1$, we define f_k as the number of subsets U of size $d+k+1$, which contain $\mathbf{0}$ in the interior[4] of their convex hull $\text{conv}(U)$ (or alternatively

[4]The interior is taken with respect to the standard topology of $\mathbf{R}^d$, i.e. the convex hull has to be full-dimensional in order to have a nonempty interior.

$\mathbf{R}^d = \text{cone}(U)$).

$$f_k = f_k(V) := |\{U \subset V : |U| = d + k + 1, \mathbf{0} \in \text{int}(\text{conv}(U))\}|.$$

The integer vector $(f_0, f_1, \dots, f_{n-d-1})$ is called the *f-vector* of V. It is closely related to the *h-vector* $(h_0, h_1, \dots, h_{n-d-1})$, which is defined by inverting the system of equations[5]

$$f_k = \sum_j \binom{j}{k} h_j, \qquad 0 \le k \le n - d - 1.$$

It is convenient to extend the range of indices to all integers by defining $f_k := h_k := 0$ for $k < 0$ or $k > n-d-1$. Furthermore, we define $g_k := h_k - h_{k-1}$. We collect the terms g_k into the *g-vector* $(g_0, g_1, \dots, g_{n-d-1}, g_{n-d})$ (note that the range of indices is larger by 1 than that of the f-vector and the h-vector). These definitions are *Gale dual* to the more common definition of f, h, and g-vectors for simplicial polytopes, see [22].

Winding numbers. Lee [12] and Welzl [22] independently observed the following geometric interpretation of the numbers g_k as winding numbers. Consider $U \subset \mathbf{R}^d$, which we think of as an affine point set. Consider the (affine) k-facets of U. Let o be a point that does not lie on any of the k-facets of U, and let ρ be a semi-infinite ray directed towards o that avoids any $(r-2)$-dimensional affine flat spanned by U. Let $g_k^+(U, \rho, o)$ be the number of k-facets that we *enter* (traverse from the positive to the negative side) as we move from infinity towards o along ρ, and let $g_k^-(U, \rho, o)$ be the number of k-facets of U that we *leave* (traverse from the negative to the positive side). It turns out that the difference $g_k^+(U, \rho, o) - g_k^-(U, \rho, o)$ is independent of the ray ρ, and coincides with $g_k(U, o)$, if we consider U as a vector configuration translated by $-o$ (thus, taking o as the origin $\mathbf{0}$).

3 Recap of the Planar Crossing Identity

We now review some the basic ingredients of the proof of the crossing identity (1). Throughout this section, let P be a set of n points in the plane.

Fact 1 *Let $k < \frac{n}{2}$ and ℓ be a line passing through p and no other point of P and ℓ^+, ℓ^- the halfplanes defined by ℓ. Denote $E_k(p)$ the set of all k-edges incident to the point $p \in P$, and let $E_k^+(p)$ and $E_k^-(p)$ denote the sets of those k-edges incident to p whose remaining endpoint lies in ℓ^+ and ℓ^-, respectively. Then the difference $|E_k(p) \cap \binom{\ell^+}{2}| - |E_k(p) \cap \binom{\ell^-}{2}|$ is*

$$\begin{array}{ll} +2 & \textit{if } |(P \setminus p) \cap \ell^+| \le k \\ -2 & \textit{if } |(P \setminus p) \cap \ell^-| \le k \\ 0 & \textit{otherwise} \end{array}$$

This directly implies a 2-dimensional exact variant of the so-called *Lovász lemma*, which is one of the basic tools to prove bounds on k-sets, both in the plane and in higher dimensions.

Fact 2 (Lovász lemma [14]) *Let ℓ be a line not passing through any point of P and ℓ^+ and ℓ^- the halfplanes defined by it. Then ℓ intersects exactly $e_k(P, \ell) := 2 \cdot \min\{k, |P \cap \ell^+|, |P \cap \ell^-|\}$ k-edges of P.*

Andrzejak et al. [4] proved the crossing identity (1) by analyzing how the quantities involved change under continuous motion of the point set, and the facts collected above are the basic ingredients that allow one to perform this analysis.

4 Crossing identity on $\mathbf{S}^2$

Our ultimate goal is to study point sets in $\mathbf{R}^3$ and find identity of a similar nature as (1), which might (or might not) help improving the upper bound on the number $e_k(P)$ in $\mathbf{R}^3$. The first step in this direction is studying vector configurations on $\mathbf{S}^2$, which are the first step from planar to three-dimensional point sets.

Let V be set of n vectors on $\mathbf{S}^2$ in general position. The first thing to observe is that the Fact 1 remains valid in this setting (instead of a line, one considers an equator on the sphere)[6]. We will simply refer to these facts, even when using their $\mathbf{S}^2$ versions. We can prove a generalization of the Lovász lemma.

Theorem 1 (Lovász lemma on $\mathbf{S}^2$) *Let ℓ be an oriented equator on $\mathbf{S}^2$ avoiding the vectors of V. Denote ℓ^+ and ℓ^- the hemisphere on the right of ℓ and the hemisphere on the left of ℓ, respectively. Then the number of (linear) k-edges intersected by ℓ is*

$$e_k(V, \ell) := 2 \cdot \big(\min\{k+1, |V \cap \ell^+|, |V \cap \ell^-|\} + g_{k-1}(V) - g_k(V) \big), \ \textit{for any } k < \frac{n}{2}. \qquad (2)$$

Note that this lemma puts in relation linear k-edges $(e_k(V, \ell))$ with k-facets of the underlying three-dimensional point set (the values $g_k(V)$ and $g_{k-1}(V)$).

4.1 Pinched crossings

As we already mentioned above, every set P of n points in the plane fulfills the identity (1):

$$\text{cr}_k(P) + \sum_{q \in P} \binom{\deg_k(q)}{2} = e_{<k}(P),$$

[5] In terms of generating functions, if we set $f(x) := \sum_k f_k x^k$ and $h(x) := \sum_k h_k x^k$, the equations yield $f(x) = h(x+1)$, i.e., $h(x) = f(x-1)$, i.e., $h_j = \sum_k (-1)^k \binom{k}{j} f_k$.

[6] The proof works in the spherical setting without a substantial change.

for every $0 \leq k < \frac{n-2}{2}$. For a point set $P \subset \mathbf{R}^3$ in convex position, the contraction P/p lies in one hemisphere and therefore is equivalent to a planar point set. Thus, summing up the identities over all $p \in P$ yields

$$\mathrm{pcr}_k(P) + 2\sum_{pq\in\binom{P}{2}} \binom{\deg_k(pq)}{2} = 3e_{<k}(P),$$

for all $0 \leq k < \frac{n-3}{2}$. We prove similar identities for edge crossings in a vector configuration on $\mathbf{S}^2$ and pinched crossings in a vector configuration on $\mathbf{S}^3$.

Theorem 2 *Let V be configuration of n vectors on the sphere $\mathbf{S}^2$, and let $0 \leq k < \frac{n-2}{2}$. Then*

$$\mathrm{cr}_k(V) + \sum_{q\in V} \binom{\deg_k(V/q)}{2} = e_{<k}(V) + m_k(V) \quad (3)$$

where

$$m_k(V) := (g_{k-1} - g_k)(2k + 1 + g_{k-1} - g_k) + 4g_{k-1}$$

and $\mathrm{cr}_k, e_{<k}$ and g_j denote $\mathrm{cr}_k(V), e_{<k}(V)$ and $g_j(V)$, respectively.

Summing these up for contractions U/p (observe, that $\sum_{p\in U} e_{<k}(U/p) = 3e_{<k}(U)$) yields:

Corollary 3 *Let U be a configuration of n vectors on the sphere $\mathbf{S}^3$, and let $0 \leq k < \frac{n-3}{2}$. Then*

$$\mathrm{pcr}_k(U) + 2\sum_{pq\in\binom{U}{2}} \binom{\deg_k(U/pq)}{2} = 3e_{<k}(U) + \sum_{p\in U} m_k(U/p) \quad (4)$$

Dey's bound $e_k \leq O(n^{4/3})$ for point sets in the plane follows by combining the upper bound $\mathrm{cr}_k = O(kn)$ with the crossing lemma of Ajtai et al. [1] and Leighton [13]. Theorem 2 immediately gives an upper bound on the number $\mathrm{cr}_k \leq n(k+1) + m_k$ (as $e_k \leq n(k+1)$ by the analysis of Clarkson and Shor [6]).

Corollary 4 *Let V be a set of n vectors on the sphere $\mathbf{S}^2$. Then $e_k(V) \leq O(\sqrt[3]{n^4 + n^2 m_k(V)})$.*

For vector sets lying on one open hemisphere (i.e. equivalent to planar point set), this coincides with the Dey's k-edge upper bound $O(n^{4/3})$ and as the m_k grows, it gets to the trivial $O(n^2)$ upper bound for the maximal possible values of m_k.

5 Concluding remarks

Using continuous motion arguments, we have derived a crossing identity for the number of pinched crossings between pairs of k-facets in $\mathbf{R}^3$. It would also be interesting to obtain an identity of a similar spirit for the number of *crossing triples* of k-facets.

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