Deflating Polygons to the Limit

Isabel Hubard* Perouz Taslakian†

Abstract

In this paper we study polygonal transformations through an operation called *deflation*. It is known that some families of polygons deflate infinitely for given deflation sequences. Here we show that every infinite deflation sequence of a polygon P has a unique limit, and that this limit is flat if and only if exactly two vertices of P move (are reflected) finitely many times in the sequence.

1 Introduction

A *deflation* of a simple polygon P is an operation that reflects a subchain of P through a line ℓ that crosses P at exactly two of its vertices such that the resulting polygon is (a) simple and (b) contains the reflected subchain within its convex hull. We call ℓ a *line of deflation*. A deflation is the inverse operation of a *pocket flip* introduced by Erdős in 1935 [5]. A *pocket* of a polygon is a maximal connected region exterior to the polygon and interior to the convex hull. A non-convex polygon has at least one pocket. A pocket flip reflects a pocket chain of P through the line incident to the edge of the pocket that is also an edge of the convex hull of P. Erdős conjectured that every polygon can be convexified after a finite number of (possibly) simultaneous pocket flips. A few years later, Nagy noted that flipping pockets simultaneously may result in self-intersecting polygons, and in 1939 he showed that every polygon is convexified after a finite number of sequential pocket flips [2]. Nagy's theorem and its proof have been rediscovered and reproved in different contexts ever since. It was only recently that Demaine et al. [4] showed that many of these proofs, including that of Nagy, are in fact either incorrect or incomplete, and provided a correct proof. Following the same spirit as pocket flips, Wegner [7] conjectured in 1993 that every polygon admits a finite number of deflations. His conjecture was disproved eight years later by Fevens et al. [6] who found a family of quadrilaterals that deflate infinitely for any deflation sequence. Ballinger [1] showed that every infinitely deflating quadrilateral is within the family described by Fevens et al., thus completing the characterization of infinitely deflating quadrilaterals. In trying to advance this characterization for general n-gons, Demaine et al. [3] show that every pentagon (having no vertices of angle 180°) has a deflation sequence that is finite. On the other hand, they show that there exist pentagons that deflate infinitely for well-chosen deflation sequences and that four of the vertices of such pentagons induce an infinitely deflating quadrilateral.

*Instituto de Matemáticas, Universidad Nacional Autónoma de México, hubard@matem.unam.mx

†Department of Computer Science, Université Libre de Bruxelles, perouz.taslakian@ulb.ac.be

As an open problem, Demaine et al. [3] ask whether every infinite deflation sequence of a polygon has a unique limit. In this paper we show that this indeed is the case. We also show that the limit of any infinite deflation sequence is flat (all vertices are collinear) if and only if exactly two of the vertices of the polygon are reflected only a finite number of times throughout the sequence.

2 Basic Notions

Using similar terminology as in [3] we let $P = \langle v_0, v_1, \dots, v_{n-1}\rangle$ be a simple polygon with a clockwise ordering of its vertices and $P^0, P^1, \dots,$ be an infinite deflation sequence of P where, for each k, $P^k = \langle v_0^k, v_1^k, \dots, v_{n-1}^k\rangle$ denotes the polygon after k arbitrary deflations. Thus, initially $P = P^0$. Throughout the paper we let $\{P^k\}$ denote the sequence $P^0, P^1, \dots$ where $k = 0, 1, 2, \dots$. Let P^* denote an accumulation point of $\{P^k\}$, when it exists, having vertices v_i^*. An infinite sequence $\{P^k\}$ has an *accumulation point* P^* if some subsequence of $\{P^k\}$ converges to P^*. We say that P^* is the *limit* of the sequence if every subsequence converges to P^*. The following proposition follows immediately from the fact that $P^{i+1} \subset \text{hull}(P^i)$, for all $i \in \mathbb{N}$.

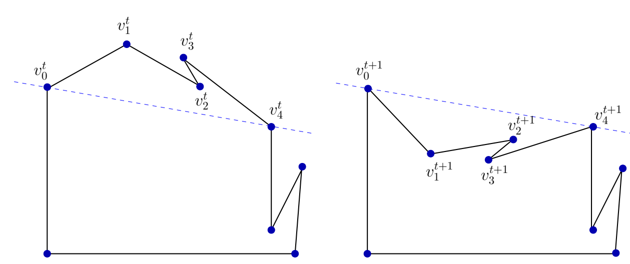

Figure 1: An example of a deflating chain: at time t, the chain $v_0^t, v_1^t, v_2^t, v_3^t, v_4^t$ gets reflected through the dotted line.

Proposition 1 *If P^* is an accumulation point of an infinite deflation sequence of a polygon P, then $P^* \subseteq \text{hull}(P^t)$ for every $t \in \mathbb{N}$.*

A *moving vertex* of an infinitely deflating pocket chain C is a vertex that is on C but not on the line of deflation of C. A *moving vertex of a sequence* $\{P^k\}$ is a vertex that is on some infinitely deflating chain of this sequence. Equivalently, a *non-moving vertex* is a vertex that is not a moving vertex, meaning it moves a finite number of times for the given deflation sequence. The *turn angle* of a vertex v_i is the signed angle $\theta \in (-180°, 180°]$ between the two vectors $v_i - v_{i-1}$ and $v_i - v_{i+1}$. A *hairpin* vertex v_i is a vertex whose incident edges overlap (forming an absolute turn angle of 180°). A *maximal infinitely deflating chain* $\mathcal{C}$ of $\{P^k\}$ is a maximal sequence of infinitely deflating pocket chains such that for every two pocket chains $C, C' \in \mathcal{C}$, there exists a sequence of infinitely deflating pocket chains $C_1, C_2, \ldots, C_k$ of P such that $C_i \in \mathcal{C}$, $C_1 = C$ and $C_k = C'$, and any two consecutive chains C_i and C_{i+1} in the sequence share at least one edge, for all $i = 1, 2, \ldots, k-1$. Every vertex on a maximal infinitely deflating chain, except the first and last along the chain, are moving vertices. Throughout the paper we assume that P does not have *straight* vertices: vertices having a turn angle of 0°. This implies that P^k has no straight vertices for any $k \in \mathbb{N}^*$, and in any accumulation point P^* [3, Corollary 1]. A *flat polygon* is a polygon with all vertices collinear. For purposes of this paper we assume that all the vertices of a flat polygon are hairpin vertices. This implies that given two consecutive vertices of a flat polygon, the remaining vertices are determined by the lengths of the edges. We hence have the following proposition.

Proposition 2 *All flat configurations of a polygon P (with no straight vertices) are equivalent (up to an isometry of the plane).*

3 The limit of an infinite deflation sequence

In this section we prove some properties of polygons that admit infinitely many deflations for some fixed infinitely deflating sequence $P = P^0, P^1, \ldots$. Note that once the sequence is fixed, there exists $t \in \mathbb{N}$ such that every line of deflation in the sequence $P^t, P^{t+1}, \ldots$ is used an infinite number of times. Abusing the notation, we rename all the polygons in the sequence as follows: $P^k := P^{k+t}$. Thus, every line of deflation in the sequence $P^0, P^1, \ldots$ is used an infinite number of times. We start by stating a lemma from [3].

Lemma 3 [Demaine et al.] *If P^* is an accumulation point of the infinite deflation sequence $P^0, P^1, P^2, \ldots$, and subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) is the pocket chain of infinitely many deflations, then $v_i^*, v_{i+1}^*, \ldots, v_j^*$ are collinear and $v_{i+1}^*, \ldots, v_{j-1}^*$ are hairpin vertices.*

Lemma 3 has important consequences in the study of infinitely deflating polygons. In particular, it implies that every infinitely deflating sequence has an accumulation point. Using Lemma 3 it is not difficult to see that two infinitely deflating pocket chains C_i and C_j of a polygon P that have at least one edge in common converge to a line in every accumulation point P^*. Thus, we have the following corollary.

Corollary 4 *Every maximal infinitely deflating chain flattens in every accumulation point.*

Lemma 3 also shows that the vertices belonging to an infinitely deflating chain become hairpin vertices at the accumulation points, and hence their absolute turn angle must decrease throughout the sequence, approaching zero. This implies that if v_i is a vertex on an infinitely deflating chain C, then v_i is on the line of deflation of some infinitely deflating chain $C' \neq C$.

In what follows we deal with infinite deflation sequences. We show that any such sequence has a limit and at least two non-moving vertices. Furthermore, if this limit is flat the sequence has exactly two non-moving vertices. To this end, we first state a fact from basic geometry in the plane, which will help us with some proofs.

Fact 5 *Let C be a circle and ℓ a line that intersects C but does not pass through its center. Thus ℓ splits C into two arcs. Let H_ℓ^+ be the open half-space determined by ℓ that contains the shorter arc C'. Then the reflection of $H_\ell^+ \cap \text{hull}(C')$ through ℓ is a subset of $\text{hull}(C)$.*

Theorem 6 *If an infinite deflation sequence of a polygon P has an accumulation point P^* that is flat, then P^* is the (unique) limit of the sequence.*

Proof. Let $\varepsilon > 0$. To show that P^* is the limit of the sequence, we shall find a time $T \in \mathbb{N}^*$ for which, $|v_i^t - v_i^*| < \varepsilon$, for every $t \geq T$ and every $i = 0, 1, \ldots n-1$. Note that P^* is flat, which means that $v_0^*, v_1^*, \ldots, v_{n-1}^*$ are collinear and, hence, have a unique configuration. Without loss of generality, assume that the vertices of P^* are along the x-axis and v_0^* and v_k^* are the vertices with the minimum and maximum x-coordinates, respectively. Let d denote the smallest distance between two vertices v_i^* and v_j^* of P^*. Because P^* is an accumulation point, there exists a subsequence $P^{t_1}, P^{t_2}, \ldots$ that converges to P^*. Hence, for $\delta < \min\{\varepsilon, \frac{d}{3}\}$, there exists a $s_\delta \in \mathbb{N}^*$ such that, for each $i = 0, 1, \ldots, n-1$, if $s > s_\delta$ then $v_i^{t_s}$ is within a disc $C_{i,\delta}$ centred at v_i^* with radius δ. Note that our choice of δ ensures that these discs do not intersect.

We first claim that a line of deflation ℓ does not cross the line segment defined by v_0^* and v_k^*. Assume otherwise: that at some time t, ℓ crosses the segment (v_0^*, v_k^*). Then the region defined by ℓ and the pocket chain to be reflected contains at least one v_i^*. If we reflect this pocket chain through ℓ, v_i^* will be left exterior to hull(P^{t+1}), a violation of Proposition 1.

Thus, for each $s > s_\delta$, we may assume that ℓ_{t_s} does not cross the segment (v_0^*, v_k^*); which impies that ℓ_{t_s} does not go through the centers of any $C_{i,\delta}$, $i = 0, \ldots, n-1$. Then ℓ_{t_s} splits each $C_{i,\delta}$ into two arcs such that every vertex to be reflected is within the region defined by ℓ_{t_s} and the shorter arc of each $C_{i,\delta}$. By Fact 5, $v_i^{t_s+1}$ remains within $C_{i,\delta}$ for all $s > s_\delta$ and $i = 0, 1, \ldots, n-1$. Note that due to our choice of δ, $C_{i,\delta} \subset C_{i,\varepsilon}$, implying that for every $t > t_{s_\delta}$, $v_i^t \in C_{i,\varepsilon}$, for every $i = 0, 1, \ldots, n-1$. Thus, P^* is the limit. □

Lemma 7 *If P deflates infinitely for a given sequence $\{P^k\}$, then $\{P^k\}$ has at least two non-moving vertices.*

Proof. Assume P has at most one non-moving vertex. Then there is a unique maximal infinitely deflating chain containing all the moving vertices of P, and by Corollary 4 this chain flattens. Therefore, every polygon with at most one non-moving vertex is flat in every accumulation point P^*. By Theorem 6, P^* is a limit.

We shall now show that the extreme vertices along this flat configuration can no longer move after a certain number of deflation steps. Because P^* is flat, without loss of generality we may assume that P^* lies on the x-axis; we can further assume that v_k^* is the vertex of P^* with largest x-coordinate. Let d denote the smallest distance between two vertices of P^*. For $\varepsilon < \frac{d}{3}$, there exists a $t_\varepsilon \in \mathbb{N}$ such that for every $t > t_\varepsilon$, v_i^t is within a disc $C_{i,\varepsilon}$ centered at v_i^* with radius ε. Note that our choice of ε ensures that these discs do not intersect.

Let L be the vertical line that goes through v_k^*, and let H_L^+ be the open half-space determined by L that contains v_i^* for all $i = 0, 1, \ldots, n-1$ (since v_k^* is the vertex of P^* with the largest x-coordinate). In particular, every $C_{i,\varepsilon}$ (and hence every v_i^t) is in H_L^+ for $i \neq k$, and $t > t_\varepsilon$ (see Figure 2). Because v_k is a moving vertex, then there must exist $t > t_\varepsilon$ such that v_k^t is reflected through some line of deflation. From the definition of deflations we know that after any deflation step the vertices on the reflected pocket chain lie inside the convex hull of the ones that did not move during that particular deflation step. In this case, as v_k^t is reflected, then v_k^{t+1} lies inside hull(P^{t+1}). We further know that $v_i^t \in C_{i,\varepsilon} \subset H_L^+$ for every $i \neq k$ and $t > t_\varepsilon$. In particular all the vertices $v_i^{t+1} \in H_L^+$. This means that hull(P^{t+1}), and consequently all the vertices of P^{t+1}, lie in H_L^+. But this implies that hull(P^{t+1}) excludes v_k^* (which lies on L) thus violating Proposition 1. Therefore, $v_k^{t_\varepsilon}$ can no longer converge to v_k^* and hence is not a moving vertex. A symmetric argument can be made about the other extreme vertex of P^*, assuming it is a moving vertex.

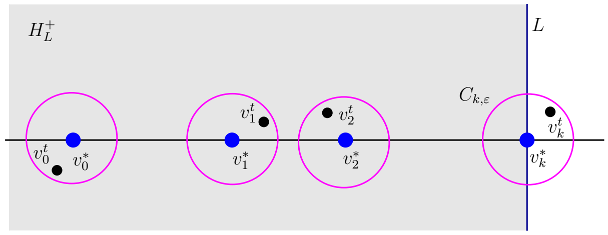

Figure 2: At step $t > T$, v_i^t lies inside $C_{i,\varepsilon}$, which in turn lies inside H_L^+ for all $i \neq k$.

Therefore, if P deflates infinitely for a given sequence, then P has at least two non-moving vertices. □

If $u_0, u_1, \ldots, u_m$ are the non-moving vertices of a deflation sequence $\{P^k\}$ of P (in clockwise order), Lemma 7 implies that all the moving vertices of the deflation sequence that lie along the chain between u_i and u_{i+1} form a maximal infinitely deflating chain, and hence, by Corollary 4, are collinear in every accumulation point. Arguing as in Theorem 6 one can show the following theorem.

Theorem 8 *If P deflates infinitely for a given deflation sequence, then this sequence has a limit defined by the non-moving vertices.*

Corollary 9 *An infinite deflation sequence of a polygon P with no straight vertices flattens in the limit if and only if it has exactly two non-moving vertices.*

Proof. If P has two non-moving vertices, then by Theorem 8 it has a limit defined by these two vertices, which in turn define a line segment. Hence P^* is flat.

Now suppose P^* is flat and has the non-moving vertices $u_0^*, u_1^*, \ldots, u_k^*$ with $k \geq 2$. Assume u_0^* and u_k^* are the extreme vertices of the limit segment and let $T \in \mathbb{N}^*$ be such that $u_0^t, u_1^t, \ldots, u_k^t$ do not move for every $t > T$. Thus we may assume that $u_i^t = u_i^*$ for all $i = 0, \ldots, k$. First note that (u_0^t, u_k^t) is not an edge of P, because otherwise P^t is nonsimple for every $t > T$. Also note that no infinite line of deflation passes through u_i^t for any $i = 1, \ldots, k-1$: any valid line of deflation through these vertices splits the polygon into two chains, each containing one of u_0^t or u_k^t; the vertices of one of these chains must move due to the deflation, thus contradicting the fact that u_0^t and u_k^t are non-moving vertices.

Let u_j^t be a non-moving vertex of P^t for some $j \in \{1, 2, \ldots, k-1\}$ such that $(u_j^t, u_{j+1}{i}^t)$ is not an edge of P^t and every vertex between u_j^t and u_{j+1}^t is moving. Note that such a j exists as P has no straight vertices. In particular, vertex v_r^t adjacent to u_j^t is moving. Let $t > T$ be such that after t deflation steps the pocket chain C^t of P^t is deflated and such that v_r^t is on C^t. This means that v_r^t moves at step t. Because u_j^t is adjacent to v_r^t, then we have two cases: either the line of deflation of P^t passes through u_j^t, which cannot happen due to the previous argument; or u_j^t is also on the deflating chain C^t. In this case u_j^t will move with the vertices of C^t to a new position contradicting the fact that it is a non-moving vertex. Therefore, both these cases are not possible, and hence u_j^t is a moving vertex for every $j = 1, \ldots, k-1$. Together with Lemma 7 we may conclude that if P^* is flat then the deflation sequence contains exactly two non-moving vertices. □

4 Polygons that deflate infinitely for every deflation sequence

In the previous section we showed that every infinite deflation sequence has a limit, and we characterized the sequences that converge to flat limits. Observe that the limit depends on the given infinite sequence, and not only on the original polygon. In fact, different deflation sequences of the same polygon may converge to different limits. In [3], Demaine et al. give examples of two polygons that deflate infinitely to a flat limit for *well-chosen* deflation sequences, but for which there always exist deflation sequences that are finite. It is not too difficult to see, for example, that the polygon in Figure 3 has at least two different infinite deflation sequences that converge to different limit points. Note however that although the polygon in this example has different limit points for different infinite deflation sequences, it also admits sequences that deflate finitely.

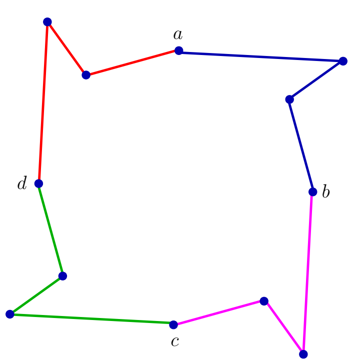

Figure 3: A polygon that deflates infinitely for well-chosen deflation sequences. Each of the chains between (a, b), (b, c), (c, d), and (d, a) induces an infinitely deflating quadrilateral. Observe that at any step if we deflate through any of the lines defined by ab, bc, cd, or da, deflations will stop for that subchain. Therefore, each infinite deflation sequence may converge to a different limit, and there exists a sequence that deflates finitely.

We complete our study of deflations by stating some facts about polygons that deflate infinitely for *every* deflation sequence. An example of such polygons is the family of quadrilaterals described by Fevens et al. [6], any deflation sequence of which has a flat limit. Given a polygon that deflates infinitely for every deflation sequence, a natural question to ask is whether or not each of these deflation sequences has the same limit, as in the case of quadrilaterals. Although we fall short of providing an answer to this question, the results of the previous section, together with arguments similar to the ones used throughout the paper can be used to obtain the following proposition regarding polygons that deflate infinitely for every deflation sequence.

Proposition 10 *Let P be a polygon that deflates infinitely for every deflation sequence.*

(a) If the limit of every deflation sequence is flat, then all these limits P^* are the same (up to an isometry of the plane) and every sequence flattens to the longest edge of the polygon.

(b) If there exists a sequence for which P^* is not flat, then there is at least one non-moving vertex in the interior of an edge of the convex hull of P^*.

References

[1] B. Ballinger. *Length-Preserving Transformations on Polygons*. Phd thesis, University of California, Davis, California, 2003.

[2] B. de Sz. Nagy. Solution of problem 3763. *American Mathematical Monthly*, 46:176–177, 1939.

[3] E. D. Demaine, M. L. Demaine, T. Fevens, A. Mesa, M. Soss, D. L. Souvaine, P. Taslakian, and G. Toussaint. Deflating the Pentagon. *Lecture Notes in Computer Science*, KyotoCGGT 2007, Kyoto, Japan, June 11-15, 2007. Revised Selected Papers, pages 52–67, 2008.

[4] E. D. Demaine, B. Gassend, J. O'Rourke, and G. T. Toussaint. All Polygons Flip Finitely... Right? In *Surveys on Discrete and Computational Geometry: Twenty Years Later*, Contemporary Mathematics, 453:231–255. American Math. Society, 2008.

[5] P. Erdős. Problem 3763. *American Mathematical Monthly*, 42:627, 1935.

[6] T. Fevens, A. Hernandez, A. Mesa, P. Morin, M. Soss, and G. Toussaint. Simple polygons with an infinite sequence of deflations. *Contributions to Algebra and Geometry*, 42(2):307–311, 2001.

[7] B. Wegner. Partial inflations of closed polygons in the plane. *Contributions to Algebra and Geometry*, 34(1):77–85, 1993.