Coloring geometric hypergraphs defined by an arrangement of half-planes

Radoslav Fulek*†

Abstract

We prove that any finite set of half-planes can be colored by two colors so that every point of the plane, which belongs to at least three half-planes in the set, is covered by half-planes of both colors. This settles a problem of Keszegh.

1 Introduction

By a *hypergraph* $H = (V, E)$ we understand a system of sets E whose elements, which are called *hyperedges*, are drawn from the set V. By a *k-coloring* of H we understand a mapping $\chi : V \to C$, where $|C| = k$. We say that an edge e is *monochromatic* under the coloring χ, if $\chi(v)$ is the same for all the vertices v in e. Then we define the *chromatic number* of H to be the minimum k such that there exists a k-coloring χ of H, which does not make any edge in E monochromatic.

Let $H = H(\mathcal{H}) = (V, E)$ denote a hypergraph having the finite set of closed half-planes $\mathcal{H}$ in $\mathbb{R}^2$ as the set of vertices, and whose hyperedges correspond to the set of points covered by at least three half-planes in $\mathcal{H}$. More formally, for each point $p \in \mathbb{R}^2$ covered by at least three half-planes in $\mathcal{H}$ the hyperedge $e_p \in E$ is the set of half-planes $\mathcal{H}$ containing p. Notice that all the points belonging to the same region in the arrangement of the lines which define half-planes in $\mathcal{H}$, correspond to the same hyperedge. Keszegh in [7] asked (in the settings of dual weak conflict-free coloring) what is the tight upper bound on the chromatic number of H. We will prove the following.

Theorem 1 *For any finite set of closed half-planes $\mathcal{H}$ the chromatic number of $H(\mathcal{H})$ is at most two. Moreover, a witnessing 2-coloring can be constructed in deterministic time $O(|V| \log |V|)$.*

The general problem of coloring hypergraphs is well studied and its investigation can be traced back to 1970's. We note that it is NP-hard to decide, whether a given hypergraph is 2-colorable. The same holds even if we restrict ourselves to 3-regular hypergraphs [6]. Hence, probably there is no nice characterization of 2-colorable hypergraphs, if we require all hyperedges to have at least three vertices, which is our case. Two well known conditions for a hypergraph H, which are easy to check, and each of which implies 2-colorability, are (1) H is balanced, (2) any union of m hyperedges contains at least $m + 1$ vertices (see e.g. [5]). However, neither of them can be applied in our case.

At the end of this section we would like to point out that one can rephrase our problem in the setting of covering decomposition, see [8]. For some recent results in the area see e.g. [9, 10]. Thus, we can say that we want to divide $\mathcal{H}$ into two parts so that any point p in the plane covered by at least three elements of $\mathcal{H}$ is covered by a half-plane in each part. Hence, the immediate consequence of Theorem 1 is the following.

Corollary 2 *Every 3-fold covering of the plane by a finite set of closed half-planes is decomposable into two parts.*

2 Preliminaries

From now on let $\mathcal{H}$ denote a finite set of closed half-planes in $\mathbb{R}^2$ in the following general position: no half-plane in $\mathcal{H}$ is defined by a vertical line, no two half-planes in $\mathcal{H}$ are defined by two parallel lines, and no three half-planes in $\mathcal{H}$ are defined by three lines intersecting in a common point. By a standard perturbation argument one can show that if the chromatic number of $H(\mathcal{H})$ is at most two, for any $\mathcal{H}$ in general position, then the same holds for any finite set of closed half-planes $\mathcal{H}'$ in $\mathbb{R}^2$.

We say that a half-plane in $\mathbb{R}^2$ is *upper* (*lower*), if it is defined as a set of points $(x, y) \in \mathbb{R}^2$ satisfying $y \le ax + b$ ($y \ge ax + b$), for some $a \ne 0, b \in \mathbb{R}$. We can partition $\mathcal{H}$ into two parts $\mathcal{H}_U$ and $\mathcal{H}_L$ containing upper and lower half-planes, respectively.

By the point-line duality in the plane we understand a transformation that takes the point $(a, b) \in \mathbb{R}^2$, $a \ne 0$, to the line $y = ax - b$ and the line $y = ax + b$, $a \ne 0$, to the point $(a, -b)$. Our duality preserves point-line incidence and above-below relationship.

By the point-line polar duality in the plane we understand a transformation that takes the line $ax + by = 1$, $(a, b) \ne (0, 0)$ to the point (a, b) and vice versa.

By the dual of a half-plane h defined by a line $y \le ax + b$ ($y \ge ax + b$) in the point-line duality we understand a vertical ray r starting at $(a, -b)$ having downward (upward) direction. This extension of the duality

*Ecole Polytechnique Fédérale de Lausanne. Email: radoslav.fulek@epfl.ch.

†This work come out of GWOP 2009 organized by Group Emo Welzl, D-INFK TI ETH Zürich, Switzerland

is natural, since a point $p \in h$, if and only if its dual line intersects r.

Let $\mathcal{R}_U$ and $\mathcal{R}_L$ denote the set containing vertical rays dual to the elements in $\mathcal{H}_U$ and $\mathcal{H}_L$, respectively. Let $\mathcal{R} = \mathcal{R}_U \cup \mathcal{R}_L$.

Using the point-line duality we can naturally recast our coloring problem so that instead of half-planes we are coloring the vertical rays in $\mathcal{R}$ and we require that any line l intersecting at least three rays in $\mathcal{R}$ intersects rays of both colors (see Figure 2).

Let $\mathcal{P}_U$ and $\mathcal{P}_L$ denote the sets of the starting points of the rays in $\mathcal{R}_U$ and $\mathcal{R}_L$, respectively. Let $\mathcal{P} = \mathcal{P}_U \cup \mathcal{P}_L$. Note that $\mathcal{P}_U$ and $\mathcal{P}_L$ could be also defined as the sets of the points dual to the lines defining the half-planes in $\mathcal{H}_U$ and $\mathcal{H}_L$, respectively. We denote by $\mathcal{P}_U^0$ and $\mathcal{P}_L^0$ the subsets of $\mathcal{P}_U$ and $\mathcal{P}_L$, respectively, containing the vertices of the upper and lower, respectively, hull of the points in $\mathcal{P}_U$ and $\mathcal{P}_L$, respectively. Having defined $\mathcal{P}_U^i$ and $\mathcal{P}_L^i$, we define $\mathcal{P}_U^{i+1}$ and $\mathcal{P}_L^{i+1}$ as the subsets of $\mathcal{P}_U \setminus \bigcup_{j \leq i} \mathcal{P}_U^j$ and $\mathcal{P}_L \setminus \bigcup_{j \leq i} \mathcal{P}_L^j$, respectively, containing the vertices of the upper and lower, respectively, hull of the remaining points.

If it does not lead to a confusion, we will be referring interchangeably to the vertices and hyperedges of H via primal or dual setting. We also refer to the vertices of H as to the elements of $\mathcal{P}$. We call a 2-coloring of the half-planes in $\mathcal{H}$ *good*, if it does not leave any region covered by at least three half-planes in $\mathcal{H}$ monochromatic, or in other words, if it witnesses that $\chi(H) = 2$.

Claim 3 *In order to prove Theorem 1, it is enough to prove it for the arrangements of half-planes $\mathcal{H}$, such that for $H = H(\mathcal{H})$ the following holds: For each $v \in V$ the intersection of all hyperedges of size 3, that contain v, is $\{v\}$.*

Let $(p_1, \ldots p_n)$ denote a sequence of points of $\mathcal{P}$. Let l denote a line in the plane. The condition $l(p_1, \ldots p_n)$ is true if and only if l intersects the vertical rays in $\mathcal{R}$ corresponding to the points $p_1, \ldots p_n$. We use the symbol $\neg$ in front of $l(p_1, \ldots p_n)$ to indicate its negation. Let $x(p)$ denote the x-coordinate of the point p. We write $p < q$ for two points in the plane if $x(p) < x(q)$. We say that a point r in $\mathcal{P}_U^i$ ($\mathcal{P}_L^i$) is between $p \in \mathcal{P}_U^i$ ($\mathcal{P}_L^i$) and $q \in \mathcal{P}_U^i$ ($\mathcal{P}_L^i$) if $p < r < q$.

Let $p_1, p_2, p_3 \ldots$ denote the points in $\mathcal{P}_U^i$ ($\mathcal{P}_L^i$) according to the order of their appearance on the hull from left to right.

The following claim allows us to consider only sets of half-planes $\mathcal{H}$ without "dummy" elements. Let p_j and p_{j+1} stand for a fixed pair of vertices in $\mathcal{P}_U^0$ ($\mathcal{P}_L^0$). In the spirit of Claim 3 we have:

Claim 4 *We can assume that there are at most three points between p_j and p_{j+1} in $\mathcal{P}_U^1$ ($\mathcal{P}_L^1$).*
Among these (at most) three points there is at most one special point p such that there exist lines l_j, l_{j+1} for which $l_j(p, p_j)$, $l_{j+1}(p, p_{j+1})$, $\neg l_j(p_{j+1})$, $\neg l_{j+1}(p_j)$, and $\neg l_j(q), \neg l_{j+1}(q)$ for all the points q, $q \neq p$, $p_j < q < p_{j+1}$ in $\mathcal{P}_U^1$ ($\mathcal{P}_L^1$). Let us call these (at most) three points by $p'_1, p = p'_2$ and p'_3, so that $p'_1 < p < p'_3$.

Moreover, if there is no special point between p_j and p_{j+1} in $\mathcal{P}_U^1$ ($\mathcal{P}_L^1$), we can assume that there is no point at all between p_j and p_{j+1}.

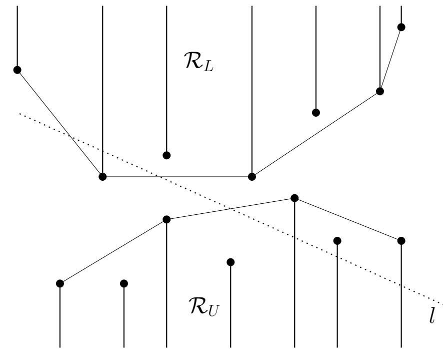

Figure 1: Dual settings

The case analysis in the main section is based on the following observation. Let U and L denote the upper and lower hull, respectively, of the finite set of points $\mathcal{P}_U$ and $\mathcal{P}_L$.

Observation 1 *If L and U intersect, then at least one of the following two sets is not empty: $P_L \cap U$, $P_U \cap L$.*

Let $p \in \mathcal{P}_U^0$, and $q \in \mathcal{P}_L^0$, $p < q$. Let l_U and r_L denote the points (if they exist) preceding and succeeding p and q, respectively, on their respective hulls. Moreover, we assume that for the line l containing l_U, p we have $l(q)$, and for the line l' containing q, r_L we have $l'(p)$. The following simple lemma is a crucial ingredient in the proof of the main theorem.

Lemma 5 *Suppose that $\mathcal{P}_U$ does not contain any point to the right of p, and $\mathcal{P}_L$ does not contain any point to the left of q. Then there exists a good 2-coloring of $\mathcal{P}$, which colors p with blue, q with red, all the vertices in $\mathcal{P}_U$ between l_U and p by red, and all the vertices in $\mathcal{P}_L$ between q and r_L by blue color. Moreover, if there is no vertex in $\mathcal{P}_U$ between l_U and p, l_U is colored by red, and if there is no vertex in $\mathcal{P}_L$ between q and r_L, r_L is colored by blue color.*

Proof. There are two cases to consider (see the upper part of Figure 2 for an illustration) according to the position of the intersection of the two lines: $l_U p$ and $q r_L$. In the figures the points are depicted by squares, discs and circles representing uncolored vertices, and vertices colored by *blue*, and *red* color, respectively. A grey area depicts a region that does not contain any point from $\mathcal{P}$ in its interior.

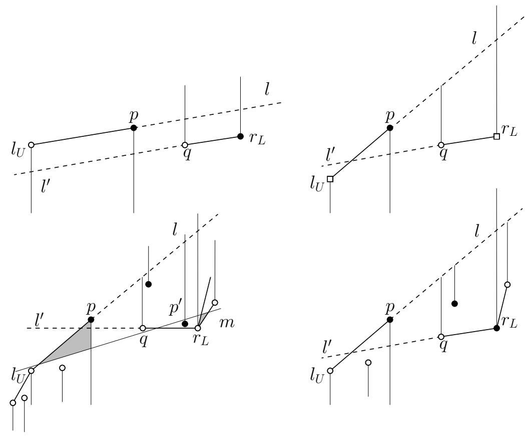

Figure 2: Lemma 5

First, assume that neither l intersects qr_L, nor l' intersects l_Up. In this case we color every point to the right of q with *blue* color and every point to the left of p with *red* color.

Hence, we can assume that l' intersects l_Up (the other case is symmetric) (see the lower part of Figure 2 for an illustration).

If there is a line m, for which $m(r_L, p, p')$ for some p', $q < p' < r_L$, $p' \in \mathcal{P}_L$, and $\neg m(p'')$ for all $p'' \in \mathcal{P}_L$, $p'' > r_L$, or $p'' \in \mathcal{P}_U$, $p'' < p$, we color l_U and r_L with *red*, the points between q and r_L with *blue* color, and the rest is colored with *red*.

Otherwise (if there is no such line m) we color l_U with *red*, r_L with *blue*, the points between q and r_L with *blue*, and the rest of the points with *red* color.

It is straightforward to check that our 2-coloring is good in every considered case, and that it satisfies the required properties. □

3 Proof of Theorem 1

First let us assume that there is point $p \in \mathbb{R}^2$ that is not covered by any half-plane in $\mathcal{H}$. Without loss of generality we can suppose that p is the origin. We use the polar duality transformation on the lines defining the half-planes in $\mathcal{H}$ thereby obtaining a set of points $\mathcal{P}_P$. Let $\mathcal{L}_P$ denote the set of line segments pp', where $p' \in \mathcal{P}_P$. Now, it is enough to two color the line segments in $\mathcal{L}_P$ so that any line intersecting at least three line segments in $\mathcal{L}_P$ intersects line segments of both colors.

Let $\mathcal{P}'$ stand for a finite set of points in the plane. We define a hypergraph $H' = (\mathcal{P}', E')$, s.t. a hyperedge in E' is the intersection of a closed half-plane with $\mathcal{P}'$ of size at least three. We use the algorithm from [7], which gives a 2-coloring of a finite set of points $\mathcal{P}'$ in the plane witnessing the fact that the chromatic number of H' is at most 2, to color the points in $\mathcal{P}_P$. A good coloring of the line segments in $\mathcal{L}_P$ is obtained by assigning to any line segment the color of its endpoint in $\mathcal{P}_P$.

Thus, we can assume that the whole plane is covered by the half-planes in $\mathcal{H}$.

Let *uphull*($\mathcal{P}_U$) and *lowhull*($\mathcal{P}_L$) denote the upper and lower hull of $\mathcal{P}_U$ and $\mathcal{P}_L$, respectively. Note, that the assumption about covering the plane by the half-planes in $\mathcal{H}$ translates in the dual setting to the assumption that uphull($\mathcal{P}_U$) and lowhull($\mathcal{P}_L$) intersect. Thus, by using Observation 1 we obtain a point $p \in \mathcal{P}_U^0$ (w.l.o.g.) contained in lowhull($\mathcal{P}_L$). Hence, we have two points $l_L, r_L \in \mathcal{P}_U^0$, $l_L < p < r_L$, such that there is no point $q \in \mathcal{P}_L^0$, for which $l_L < q < r_L$. Let v denote a vertical line through p. W.l.o.g we can assume that if p is the leftmost point in $\mathcal{P}_U$, $|\mathcal{P}_U| = 1$.

If $|\mathcal{P}_U| > 1$, let h denote the line through p, which is the extension of the side of uphull($\mathcal{P}_U$) ending at p (if we traverse the hull from the left to right). Let l_U denote the other endpoint of this side. Let r_U denote the point following p on the upper hull (if it exists). The lines v and h divide the plane into 4 regions (see Figure **??**). Depending on the containment of l_L and r_L in these 4 regions, on the existence of an intersection between segments l_Lr_L and l_Up, and on whether $l_L < l_U$ holds, we distinguish the following 4 cases.

In each of the cases below we define a good 2-coloring χ of H according to a certain rule which we leave out in this extended abstract.

a) In this case we have: r_L is above h, which implies that l_Up and l_Lr_L do not intersect each other.

b) In this case we have: r_L is below h, l_Up and l_Lr_L do not intersect each other, and $l_L < l_U$.

c) In this case we have: r_L is below h, and l_Up and l_Lr_L intersect.

d) In this case we have: r_L is below h, l_Up and l_Lr_L do not intersect each other, and $l_L > l_U$.

Note that if $|\mathcal{P}_U| = 1$, the situation can be handled as a special case of case (c).

As for the algorithmic part of the statement. It is easy to see that the above proof can be turned into an algorithm which 2-colors H. Moreover, it is also easy to see that the bottle necks of the algorithm are constructing a convex hull (see e.g [12]), sorting the points in $\mathcal{P}$ according to the x-coordinate, and running the algorithm from [7] (in case when there is an uncovered point of the plane). Since each of these operations requires the claimed running time, and each of them is carried out constant number of times, the rest of the theorem follows.

4 Discussion

The problem we consider in this paper was originally stated in the setting of conflict-free coloring defined as follows[1].

Let P denote a finite set of points in $\mathbb{R}^2$. Let $\mathcal{R}$ denote a set (possibly infinite) of regions (subsets of $\mathbb{R}^2$). A conflict-free coloring of P with respect to $\mathcal{R}$ is an assignment χ of colors $\{1, \ldots k\}$ to the points in P, such that for any range $r \in \mathcal{R}$, the set $P' := P \cap r$ contains a point $p \in P'$ of unique color $c \in \{1, \ldots k\}$, i.e. for all $p' \in P'$, s.t. $p' \neq p$, we have $\chi(p) \neq \chi(p')$. For a weak conflict-free coloring we only require that P' is not monochromatic, i.e. if $|P'| > 1$, P' contains two different points p, q such that $\chi(p) \neq \chi(q)$.

In the dual version of conflict-free coloring instead of finite set of points P we fix a finite set of regions $\mathcal{R}$ (subsets of $\mathbb{R}^2$). A conflict-free coloring of $\mathcal{R}$ with respect to a set of points $P \subseteq \mathbb{R}^2$ is an assignment χ of colors $\{1, \ldots k\}$ to the regions in $\mathcal{R}$, so that any point $p \in P$ covered by at least one region in $\mathcal{R}$ is covered by a region in $\mathcal{R}$ of unique color (in the same sense as above). Similarly, in the weak version of dual conflict-free coloring we require that if $p \in P$ is covered by more than two regions in $\mathcal{R}$ not all of them have the same color.

In each of the above cases, chromatic number is defined as the minimum number of colors needed to obtain the desired coloring.

Thus, our problem can be stated as a problem of estimating chromatic number in the setting of dual weak conflict free coloring with respect to the set of closed half-planes.

Since the notion of conflict-free coloring was introduced, many variants of the problem of estimating the conflict-free chromatic number and its dual has been considered, e.g. the instances of the problem where the set of regions consists of discs [3, 13], rectangles [1, 4, 11]. A generalization of our problem was recently considered in [2, 14]. In fact, we strengthen a bit their result in one special case, i.e. we proved that $p_{\tilde{\mathcal{H}}}(2) = 3$[2] (as defined there). Before, it was known that $p_{\tilde{\mathcal{H}}}(2) \leq 4$ ([7, 13]). It would be interesting to see whether the ideas from our proof can be generalized and used to strengthen Corollary 1 from [2].

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[1]We decided to formulate it as a hypergraph coloring problem, because we found it more natural.

[2]$p_{\tilde{\mathcal{H}}}(k)$ is defined as the minimum number l so that we can k-color any finite set of half-planes such that any region covered by at least l half-planes is covered by half-planes of all k colors.

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