Triangulations with many points of even degree

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Abstract

Let S be a set of points in the plane in general position. A triangulation of S will be called *even* if all the points of S have an even degree. We show how to construct a triangulation of S containing at least $\lfloor \frac{2n}{3} \rfloor - 3$ points with even degree; this improves slightly the bound of $\lceil \frac{2(n-1)}{3} \rceil - 6$ by Aichholzer et. al. [1]. Our proof can be easily adapted to give, through a long case analysis, triangulations with $\lfloor \frac{4n}{5} \rfloor - c$ vertices with even degree.

1 Introduction

Let S be a set of n points on the plane in general position, and let $\mathbf{Conv}(S)$ denote the convex hull of S. A triangulation of S is a plane graph G whose vertex set is S, and having $2n + i - 3$ edges, where i is the number of elements of S in the interior of $\mathbf{Conv}(S)$. A triangulation of S is called *even* if all the vertices of S have an even degree.

Even triangulations are used in several problems. Our original motivation to study them, arises from applications of them to several Art Gallery problems [6]. In particular Hoffman and Kriegel proved that every 2-connected bipartite plane graph can always be completed to an even triangulation [3]. Combined with Whitney's theorem this result implies that a plane triangulation is 3-colorable if and only if all of its vertices have an even degree, see Lováz for a nice proof of this result [4]. Hoffman and Kriegel then used the previous result to prove that any orthogonal polygon with holes, can always be guarded with at most $\lfloor \frac{n}{3} \rfloor$ vertex guards [3].

In a different setting, while studying the existence of monochromatic empty quadrilaterals, Aichholzer et. al. [2] obtained some results regarding the existence of triangulations of point sets S, such that the degrees of the vertices of the triangulations satisfy some parity constrains imposed in advance on the elements of S. They proved that for a given parity assignment to the elements of S, there is always a triangulation that satisfies approximately half of these constrains. That result was later improved in 2009 by Aichholzer et. al. [1] to $\lceil \frac{2(n-1)}{3} \rceil - 6$. In this paper we give a new proof of this result. Our proof can be easily extended (with a long case analysis) to prove that any set of n points in general position always has a triangulation with at least $\lfloor \frac{4n}{5} \rfloor - c$ even degree vertices, c a constant. In what follows, S will always denote a set of n points on the plane in general position, $n \geq 3$.

2 A triangulation with $\lfloor \frac{2n}{3} \rfloor - 3$ even vertices.

We will prove the following theorem:

Theorem 1 *For any set S of n points on the plane in general position, there is a triangulation such that at least $\lfloor \frac{2n}{3} \rfloor - 3$ elements of S have even degree.*

Our proof is constructive; given a set S of n points, we show how to construct a triangulation of S with $\lfloor \frac{2n}{3} \rfloor - 3$ points with even degree.

Suppose that there is a unique element p_0 of S with the lowest y-coordinate. Order the elements of $S - p_0$ radially around it. Split the elements of $S - p_0$ into groups $S_1, S_2, \ldots$ of four elements each (except perhaps for the last subset), according to their order around p_0, and calculate the $\mathbf{Conv}(S_1 + p_0), \mathbf{Conv}(S_2 + p_0), \ldots$; the edges of these convex hulls will belong to the final triangulation, (see Figure 1).

We then triangulate the region $\mathbf{Conv}(S) - \{\mathbf{Conv}(S_1 + p_0) \cup \mathbf{Conv}(S_2 + p_0), \ldots\}$ any which way, obtaining a geometric graph G_0 on S. Each point on the boundary of the union of our slices is labelled with $\oplus$ or $\ominus$ if they have even or odd degree in G_0.

Next we process the points in $S_1, S_2, \ldots$, from left to right in such a way that when an S_i is processed, two of the first three elements in it (according to their radial order around p_0) have even degree. This will prove our result. Suppose then that we have already

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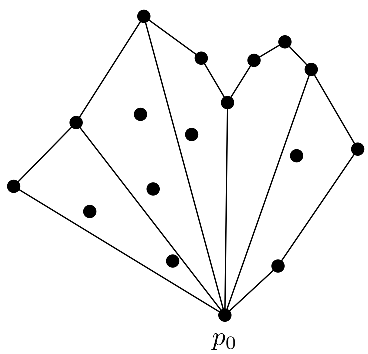

Figure 1: Slices.

processed $S_1, \ldots, S_{k-1}$. We show now how to process S_k.

Let $\mathbf{Uconv}(S_k)$ be the *upper convex hull* of S_k, that is the path formed by the vertices on $\mathbf{Conv}(S_k) - p_0$. Three cases arise according to the size of $\mathbf{Uconv}(S_k)$; it has four, three, or two vertices (see Figure 2).

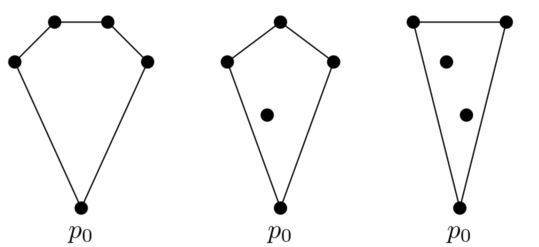

Figure 2: Possible upper convex hulls.

Case 1: Suppose first that $\mathbf{Uconv}(S_k)$ has four vertices labelled $p_1, \ldots, p_4$. There are $2^4 = 16$ different possible possibilities for the degree parities of the vertices of S_k in G_0. Since we will only fix the parities of p_1, p_2, and p_3, we can ignore the parity of p_4, it will be taken care of when we process the next slice. Thus we just have to deal with only $2^3 = 8$ possibilities.

$$\begin{array}{cc} \ominus\ominus\ominus\circledast & \oplus\ominus\ominus\circledast \\ \ominus\ominus\oplus\circledast & \oplus\ominus\oplus\circledast \\ \ominus\oplus\ominus\circledast & \oplus\oplus\ominus\circledast \\ \ominus\oplus\oplus\circledast & \oplus\oplus\oplus\circledast \end{array}$$

Observe that if we complete a triangulation of the interior of $\mathbf{Conv}(S_k)$ of S_k by joining p_0 to p_2 and p_3, the parities of p_2 and p_3 will change, (see Figure 3). If the parities of these vertices were $\ominus\ominus\ominus\circledast$, $\oplus\ominus\ominus\circledast$, $\oplus\ominus\oplus\circledast$ or $\oplus\oplus\ominus\circledast$, two of p_1, p_2, p_3 would end up with even parity.

If instead we connect p_4 to p_1 and p_2, only the parities of p_1 and p_2 will change (see Figure 4). This takes care of cases $\ominus\ominus\oplus\circledast$ and $\ominus\oplus\oplus\circledast$.

Finally, if we connect p_1 to p_3 and p_4 the parity changes in these vertices would solve cases $\ominus\oplus\ominus\circledast$ and $\oplus\oplus\oplus\circledast$, see Figure 5.

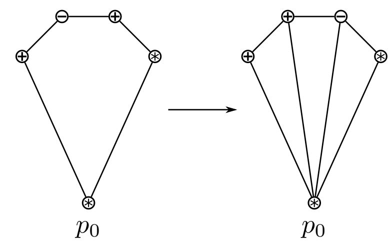

Figure 3: Changing the parity of p_2 and p_3.

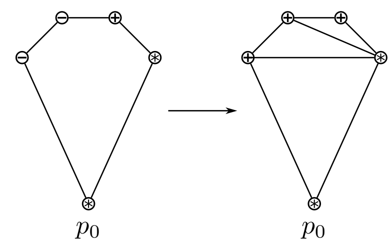

Figure 4: Changing the parity of p_1 and p_2.

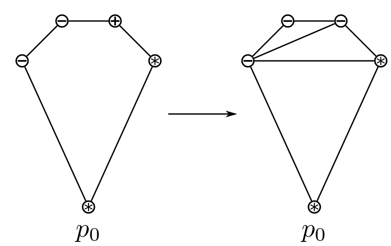

Figure 5: Changing the parity of p_3.

Case 2: Suppose next that $\mathbf{Uconv}(S_k)$ has three vertices labelled p_1, p_2, p_3. In this case, we only have $2^2 = 4$ possibilities for the degrees parities of p_1 and p_2, namely:

$$\begin{array}{cc} \ominus\ominus\circledast & \oplus\ominus\circledast \\ \ominus\oplus\circledast & \oplus\oplus\circledast \end{array}$$

If we connect the remaining point, say p, of S_k in the interior of $\mathbf{Conv}(S_k)$ to p_0, p_1, p_2, and p_3, the parities of p_1 and p_2 change, and p ends with degree four. This solves cases $\ominus\ominus\circledast$, $\oplus\ominus\circledast$ and $\ominus\oplus\circledast$ (see Figure 6).

The case $\oplus\oplus\circledast$ is harder to solve. Let ℓ be the line passing through p_1 and p_3. Two possibilities arise: p is below, or above ℓ. The first case is solved triangulating the interior of S_k as shown in Figure 7(a). For the second case, two more sub-cases arise: p lies to the right or to left of the line joining p_0 to p_2. In the first sub-case we triangulate as in Figure 7(b).

The second sub-case is harder to solve, and will be dealt with in Section 3.

Case 3: Suppose that $\mathbf{Uconv}(S_k)$ has two vertices labelled p_1, p_2. We have only two possibilities for the parity of p_1, $\ominus$ or $\oplus$. Let p and q be the elements of S_k in the interior of $\mathbf{Conv}(S_k)$. If the line through

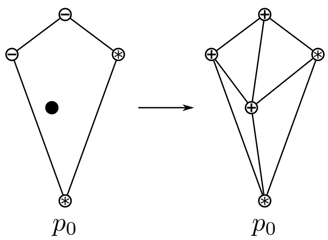

Figure 6: Changing the parity of p_1, p_2 with even degree p.

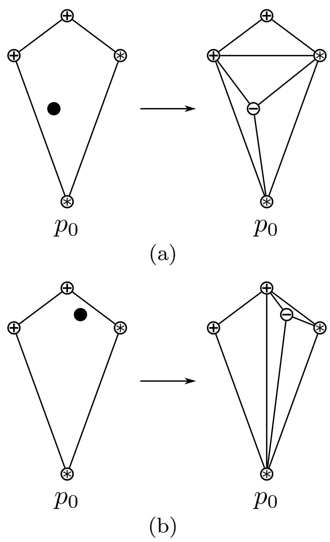

Figure 7: Solution to two of the three possibilities when we have the case $\oplus \oplus \circledast$.

p and q intersects the line segments joining p_0 to p_2, and p_1 to p_2, then triangulate the interior of S_k as in Figure 8 or Figure 9, according to the parity of the degree of p_1.

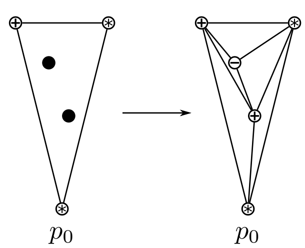

Figure 8: Case $\oplus$.

A similar solution applies when the line through p and q intersects the line segments joining p_0 to p_1 and p_2. The last, and harder case, is when the line through p and q intersects the line segments joining p_1 to p_0 and p_2. This case is again solved in Section 3.

This concludes the proof of Theorem 1.

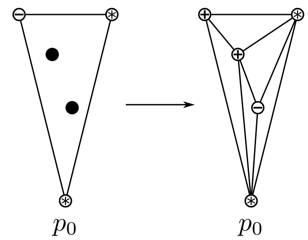

Figure 9: Case $\ominus$.

3 The bad cases

Two cases remain to be solved, those depicted in Figure 10. We only outline how to solve these cases, as their complete solution involves a long and unenlightening case analysis. A complete list of all cases to solve and to ensure that any set of points has $\lfloor \frac{2n}{3} \rfloor - 3$ points have an even degree is available online at [5].

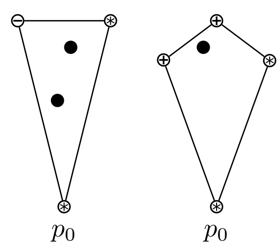

Figure 10: The bad cases.

To solve these cases, we proceed as follows: If while finding the subsets $S_1, \ldots$ we detect an S_j belonging to either of our bad cases, we modify our subsets as follows: We join S_j with S_{j+1}, and solve instead for $S_j \cup S_{j+1}$. Notice that this will change the region $\mathbf{Conv}(S) - \{\mathbf{Conv}(S_1 + p_0) \cup \mathbf{Conv}(S_2 + p_0), \ldots\}$ to be triangulated.

Observe that $\mathbf{Uconv}(S_j \cup S_{j+1} + p_0)$ may have up to six vertices. As before, we want to triangulate the interior of $S_j \cup S_{j+1} + p_0$ such that at least four out of the first six vertices of $S_j \cup S_{j+1}$ have even degree. As before, the last vertex of $\mathbf{Uconv}(S_j \cup S_{j+1} + p_0)$ will be taken care of in the next slice. We show how this can be done in a concrete example in Figure 11; the remaining cases are available at [5].

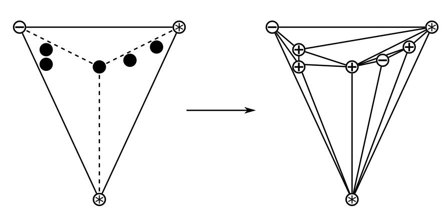

Figure 11: A bad case solved.

4 Conclusion

We proved that for any point set S on the plane in general position, can be triangulated in such a way that the number of vertices with even degree in our triangulation is at least $\lfloor \frac{2n}{3} \rfloor - 3$. The constant arises when our last slice is a bad slice, or has less than four vertices. We point out that using a long, tiring, and unenlightening case analysis, our method can be easily to the case when each S_k has six elements (plus p_0). This yields triangulations of S with at least $\lfloor \frac{4n}{5} \rfloor - c$ points with even degree. In fact, we believe that if we were to consider subsets S_i of S with more elements, and perform a huge case analysis, improvements on the bounds obtained here would arise. An interesting open problem is that of finding a different, shorter, and simpler proof of our results. In fact, we believe that the next conjecture, posed first in [2] is true:

Conjecture 1 *For any set of n points S in general position, there always exists a triangulation in which $n - o(n)$ elements of S have even degree.*

In fact, we believe that there is a triangulation of S in which, all but a constant number of elements of S have even degree.

We point out that our proof easily adapts to solve the more general *Parity Constraints Problem* introduced in [1]. In this problem we assign to each element of S a parity. In [1] the prove that for any given parity assignment to the elements of S, there is always a triangulation of S that satisfies at least $\lceil \frac{2(n-1)}{3} \rceil - 6$ parities. By manipulating properly the parity assignment to the elements of S, our results yield the same values obtained here for even triangulations.

References

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