Maximum Geodesic Routing in the Plane with Obstacles

Matthew O'Meara David L. Millman Jack Snoeyink Vishal Verma
Department of Computer Science, University of North Carolina at Chapel Hill*

Abstract

Do convex obstacles in the plane always leave 3 separate escape routes? Here, an escape route is a locally geodesic path that avoids the obstacles; escape routes are separate if they have no point in common but their origin. We answer this question, posed at FWCG '09 by Al-Jubeh, Ishaque and Tóth, in the affirmative and show how to efficiently compute the routes.

1 Introduction

Given a set of obstacles in the plane and a fixed point, how may separate geodesic paths out beyond all obstacles are there? Arkin *et al.* [4] observed that there are always two monotone paths among convex obstacles, which can be turned into separate geodesics [2, 8]. It is relatively easy to construct examples like Figure 1 that do not have four paths. At the 2009 Fall Workshop in Computational Geometry, Al-Jubeh, Ishaque and Tóth asked, "does every configuration of disjoint convex obstacles always leave 3 separate geodesic paths out?" With co-authors [1, 3] they have several results on vertex disjoint paths in embedded graphs, but wanted separate geodesics. We answer their question in the affirmative.

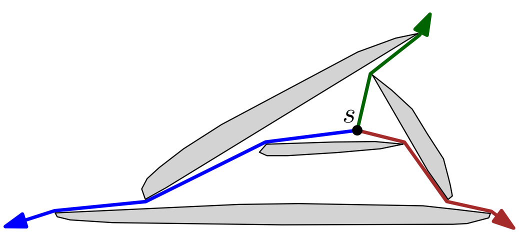

Figure 1: Disjoint convex obstacles and a starting point s that admit only 3 separate geodesic paths out.

Let s be a chosen starting point and t be the point at infinity. Let O denote a fixed set of disjoint, convex, polygonal obstacles in the plane, ∂O its boundary and $\overline{O}$ its complement. Our approach, defined precisely in the next section, is to partition $\overline{O}$ into a *radial trapezoidation* T by drawing segments of lines through s. We add enough segments that the dual graph $dual(T)$ has maximum degree 3. We show that the min-cut separating s from t in $dual(T)$ has 3 edges, which means that a max-flow algorithm will find 3 edge-disjoint paths in $dual(T)$. Since the dual graph has maximum degree 3, the paths must be vertex disjoint. Vertex disjoint paths in the dual graph can be pulled tight to become separate geodesic paths in $\overline{O}$.

* [momeara,dave,snoeyink,verma]@cs.unc.edu

2 Radial trapezoidation and min-cut

Consider the lines through s that are tangent to any convex obstacle in O. The radial trapezoidation T of $\overline{O}$ is formed by drawing the segments from each point of tangency toward and away from s until encountering the first obstacle or s itself. Figure 2 shows an example. Even though the added radial segments are not parallel (unless we do a projective transformation sending s to infinity) we still call the regions trapezoids. Each trapezoid has a top obstacle or is unbounded, a bottom obstacle or s, and points of tangency defining 2 lines that contain 2 to 4 radial segments. Split each trapezoid bounded by 4 radial segments (i.e., each trapezoid where neither tangent point defining the radial sides is on the top or bottom obstacles) by adding a new radial segment. Figure 2 depicts three such segments with dashed lines.

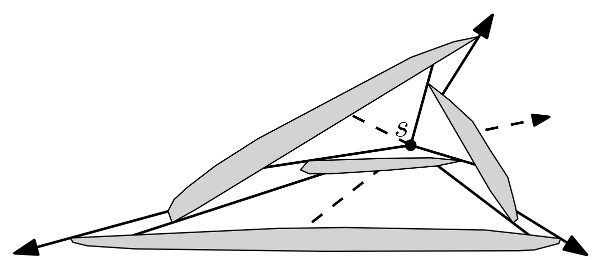

Figure 2: The radial trapezoidation has tangent segments (solid) and splitting segments (dashed) on lines through s.

The *dual graph* of a partition T of $\overline{O}$, denoted $dual(T)$, is an abstract graph in which the vertices are the cells of the partition and the edges are the boundaries between the cells. We add s and t as vertices to the dual graph and edges from s to each cell incident on s and from t to each unbounded cell. We say that the points s and t are k*-edge connected* if at least k edges of $dual(T)$ must be removed to disconnect s and t, and k*-vertex connected* if k vertices of $dual(T)$ must be removed to disconnect s and t.

Lemma 1 *Given disjoint, convex, polygonal obstacles in the plane with a total of n vertices, a radial trapezoidation can be built in $O(n \log n)$ time, or $\Theta(n)$ time if a triangulation of $\overline{O}$ is given in the input. Its dual graph has maximum degree 3.*

Proof. The radial trapezoidation T with respect to a point s can be built by a radial sweep, by randomized incremental construction, or by converting from a triangulation; we need only minor modifications of algorithms for standard trapezoidations (aka vertical visibility maps) [6, 7]. Trapezoids with degree 4 in the dual can then be split in $O(1)$ time apiece. □

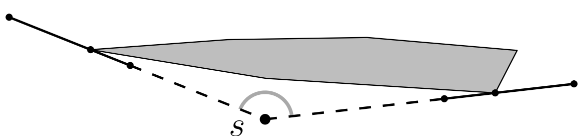

Figure 3: To show there are no separating 2-cycles, note that any convex obstacle subtends an arc of $< \pi$ radians.

Lemma 2 *For a radial trapezoidation T, in the graph $dual(T)$ the nodes for s and t are 3-edge connected.*

Proof. The obstacles in O are disjoint, so each k-edge cut in $dual(T)$ can be identified with an alternating cycle of k boundary edges of T and at most k obstacles surrounding s. Each radial boundary edge subtends 0 radians. Each obstacle is convex, so, as in Figure 3, it subtends less than π radians, which is less than half of the total 2π required. Hence $k \geq 3$. □

Lemma 3 *Consider any partition of $\overline{O}$ with all vertices on ∂O and straight boundary edges. Any path P from s to t visits cells of the partition; the geodesic path from s to t homotopic to P visits a subset of these cells.*

Proof. This is best proved using a universal cover, which lifts the possible paths to a simply connected space (see [8, Theorem 3.2]). We sketch the key idea: Let R be the union of cells visited by P and consider a partition edge e on the boundary of R. Since e is a straight line with end points affixed to obstacles and the end points of P are fixed, any continuous deformation of P that crosses e does so an even number of times, and can be shortened to follow e. Thus the geodesic path homotopic to P is contained in R. □

Theorem 4 *For any set of disjoint convex polygonal obstacles in the plane O and a starting point $s \in \overline{O}$, there are 3 disjoint geodesic paths from s to t. Paths can be found in $\Theta(n)$ time from the radial trapezoidation.*

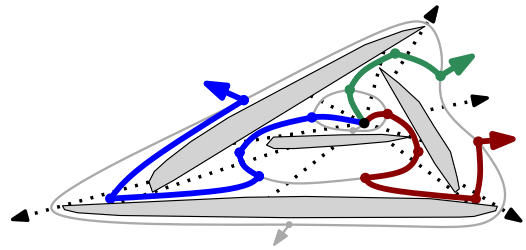

Figure 4: The dual graph of the radial trapezoidation with highlighted vertex disjoint paths from s to t.

Proof. Given O, a starting point s and a radial trapezoidation T of $\overline{O}$, by lemma 1, s and t are 3-edge connected. By applying the min-cut max-flow theorem [5], there are 3-edge disjoint paths from s to t in $dual(T)$. Figure 4 is an example. Three paths can be found with a depth first search and 3 Ford-Fulkerson path augmentations [5], taking $O(n)$ time. Since the maximum degree for any vertex in $dual(T)$ is 3, edge disjoint paths are also vertex disjoint paths, which can be embedded into T as paths that pass through disjoint trapezoids. By lemma 3, the paths can be simultaneously pulled to geodesics while remaining separate. Since O is convex and polygonal, finding the geodesic paths in $dual(T)$ reduces to finding the vertices that the paths turn on [8]. Thus, the paths can be found in $\Theta(n)$ time from the radial trapezoidation. □

References

[1] M. Al-Jubeh, G. Barequet, M. Isahique, D. L. Souvaine, C. Toth, and A. Winslow. Connecting obstacles in vertex-disjoint paths. In *26th European Workshop Comp Geom*, Dortmund, Germany, 2010.

[2] M. Al-Jubeh, M. Hoffmann, M. Ishaque, D. Souvaine, and C. Toth. Convex partitions with 2-edge connected dual graphs. *J Comb Opt*, 2010. Preprint `www.eecs.tufts.edu/~mishaq01/JOCO-2010.pdf`.

[3] M. Al-Jubeh, M. Isahique, K. Redei, D. L. Souvaine, and C. Toth. Tri-edge-connectivity augmentation for straight line graphs. In *20th ISAAC*, 2009.

[4] E. M. Arkin, R. Connelly, and J. S. Mitchell. On monotone paths among obstacles with applications to planning assemblies. In *5th ACM SCG*, pages 334–343, 1989.

[5] T. Cormen, C. Leiserson, L. Rivest, and C. Stein. *Introduction to Algorithms.* MIT Press, Cambridge, 2nd edition, 2003.

[6] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications.* Springer, 2008.

[7] A. Fournier and D. Y. Montuno. Triangulating simple polygons and equivalent problems. *ACM Trans. Graph.*, 3(2):153–174, 1984.

[8] J. Hershberger and J. Snoeyink. Computing minimum length paths of a given homotopy class. *Comp Geom: Theory Appl.*, 4(2):63–97, 1994.