Approximate Euclidean Ramsey theorems

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Abstract

According to a classical result of Szemerédi, every dense subset of $1, 2, \ldots, N$ contains an arbitrary long arithmetic progression, if N is large enough. Its analogue in higher dimensions due to Fürstenberg and Katznelson says that every dense subset of $\{1, 2, \ldots, N\}^d$ contains an arbitrary large grid, if N is large enough. Here we present geometric variants of these results for separated point sets on the line and respectively in the Euclidean space: (i) every dense separated set of points in some interval $[0, L]$ on the line contains an arbitrary long approximate arithmetic progression, if L is large enough. (ii) every dense separated set of points in the d-dimensional cube $[0, L]^d$ in $\mathbb{R}^d$ contains an arbitrary large approximate grid, if L is large enough. A further generalization for any finite pattern in $\mathbb{R}^d$ is also established. The separation condition is shown to be necessary for such results to hold. In the end we show that every sufficiently large point set in $\mathbb{R}^d$ contains an arbitrarily large subset of almost collinear points. No separation condition is needed in this case.

Keywords: Euclidean Ramsey theory, approximate arithmetic progression, approximate homothetic copy, almost collinear points.

1 Introduction

Let us start by recalling the classical result of Ramsey from 1930:

Theorem 1 (Ramsey [23]). *Let $p \leq q$, and r be positive integers. Then there exists a positive integer $N = N(p, q, r)$ with the following property: If X is a set with N elements, for any r-coloring of the p-element subsets of X, there exists a subset Y of X with at least q elements such that all p-element subsets of Y have the same color.*

As noted in [4], perhaps the first Ramsey type result of a geometric nature is Van der Waerden's theorem on arithmetic progressions:

Theorem 2 (Van der Waerden [26]). *For every positive integers k and r, there exists a positive integer $W = W(k, r)$ with the following property: For every r-coloring of the integers $1, 2, \ldots, W$ there is a monochromatic arithmetic progression of k terms.*

As early as 1936, Erdős and Turán have suggested that a stronger *density* statement must hold. Only in 1975, Szemerédi succeeded to confirm this belief with his celebrated result:

Theorem 3 (Szemerédi [25]). *For every positive integer k and every $c > 0$, there exists $N = N(k, c)$ such that every subset X of $\{1, 2, \ldots, N\}$ of size at least cN contains an arithmetic progression with k terms.*

This is a fundamental result with relations to many areas in mathematics. Szemerédi's proof is very complicated and is regarded as a mathematical tour de force in combinatorial reasoning [18, 22]. Another proof of this result was obtained by means of ergodic theory by Fürstenberg [8] in 1977.

A homothetic copy of $\{1, 2, \ldots, k\}^d$ is also called a *k-grid* in $\mathbb{R}^d$. The following generalization of Van der Waerden's theorem to higher dimensions is given by the Gallai–Witt theorem [18, 22]:

Theorem 4 (Gallai–Witt [22]). *For every positive integers d, k and r, there exists a positive integer $N = N(d, k, r)$ with the following property: For every r-coloring of the integer lattice points in $\{1, 2, \ldots, N\}^d$, there exists a monochromatic homothetic copy of $\{1, 2, \ldots, k\}^d$. More precisely, there exist $(a_1, a_2, \ldots, a_d) \in \{1, 2, \ldots, N\}^d$, and a positive integer x such that all points of the form $(a_1 + i_1 x, a_2 + i_2 x, \ldots, a_d + i_d x)$, $\quad i_1, i_2, \ldots, i_d \in \{0, 1, \ldots, k-1\}$ are of the same color.*

A higher dimensional generalization of Szemerédi's density theorem was obtained by Fürstenberg and Katznelson [9]; see also [22].

Theorem 5 (Fürstenberg–Katznelson [9]). *For every positive integers d, k and every $c > 0$, there exists a positive integer $N = N(d, k, c)$ with the following property: every subset X of $\{1, 2, \ldots, N\}^d$ of size at least cN^d contains a homothetic copy of $\{1, 2, \ldots, k\}^d$.*

The proof of Fürstenberg and Katznelson uses infinitary methods in ergodic theory. As noted in [22], no combinatorial proof is known.

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In the first part of our paper (Section 2), we present analogues of Theorems 2, 3, 4, and 5, for point sets in the Euclidean space. Specifically, we obtain (restricted) Ramsey theorems for *separated* point sets, for finding approximate homothetic copies of an arithmetic progression on the line and respectively of a grid in $\mathbb{R}^d$. The latter result carries over for any finite pattern point set and every dense and sufficiently large separated point set in $\mathbb{R}^d$. It is worth noting that the separation condition is necessary for such results to hold (Proposition 1 in Section 2). While for Theorems 2, 3, 4, and 5, the separation condition comes for free for any set of integers, it has to be explicitly enforced for point sets.

The exact statements of our results (Theorems 6, 7 and 8) are to be found in Section 2 following the definitions. Fortunately, the proofs of these theorems are much simpler than of their exact counterparts previously mentioned. Moreover, the resulting upper bounds are much better than those one would get from the integer theorems. The proofs are constructive and yield very simple algorithms for computing the respective approximate homothetic copies given input point sets satisfying the requirements.

In the second part (Section 3), we present an unrestricted theorem (Theorem 9) which shows the existence of an arbitrary large subset of almost collinear points in every sufficiently large point set in $\mathbb{R}^d$. No separation condition is needed in this result.[1]

Applications. Many other Ramsey type problems in the Euclidean space have been investigated in a series of papers by Erdős et al. [4, 5, 6] in the early 1970s, and later by Graham [10, 11, 12, 13, 14]. Van der Waerden's theorem on arithmetic progressions has inspired new connections and numerous results in number theory, combinatorics, and combinatorial geometry [1, 2, 7, 10, 15, 16, 17, 18, 19, 20, 22, 24], where we only named a few here.

Our analogues of Theorems 2, 3, 4, and 5, for point sets in the Euclidean space may also find fruitful applications in combinatorial and computational geometry. It is obvious that general point sets are much more common in these areas than the rather special integer or lattice point sets that occur in number theory and integer combinatorics. A first application needs to be mentioned: A result similar to our Theorem 6 has been proved instrumental in settling a conjecture of Mitchell [21] on illumination for maximal unit disk packings: It is shown [2] that any dense (circular) forest with congruent unit trees that is deep enough has a hidden point. The result that is needed there is an approximate equidistribution lemma for separated points on the line, which is a relaxed version of our Theorem 6.

[1]Due to space limitations, proofs have been omitted from this abstract.

2 Approximate homothetic copies of any pattern

Definitions. Let $\delta > 0$. A point set S in $\mathbb{R}^d$ is said to be *δ-separated* if the minimum pairwise distance among points in S is at least δ. For two points $p, q \in \mathbb{R}^d$, let $d(p, q)$ denote the Euclidean distance between them. The closed ball of radius r in $\mathbb{R}^d$ centered at point $z = (z_1, \ldots, z_d)$ is $B_d(z, r) = \{x \in \mathbb{R}^d \mid d(z, x) \leq r\} = \{(x_1, \ldots, x_d) \mid \sum_{i=1}^{d}(x_i - z_i)^2 \leq r^2\}$.

Given a point set (or "pattern") $P = \{p_1, \ldots, p_k\}$ of k points in $\mathbb{R}^d$ and another point set Q with k points: (i) Q is *similar* to P, if it is a magnified/shrunk and possibly rotated copy of P. (ii) Q is *homothetic* to P, if it is a magnified/shrunk copy of P in the same position (with no rotations).

Approximate similar copies and approximate homothetic copies are defined as follows. See also Fig. 1 for an illustration. Given point sets P and Q as above and $0 < \varepsilon \leq 1/3$:

- Q is an *ε-approximate similar* copy of of P, if there exists Q' so that Q' is similar to P, and each point $q'_i \in Q'$ contains a (distinct) point $q_i \in Q$ in the ball of radius εd centered at q'_i, where d is the minimum pairwise distance among points in Q'.

- Q is an *ε-approximate homothetic* copy of of P, if there exists Q' so that Q' is homothetic to P, and each point $q'_i \in Q'$ contains a (distinct) point $q_i \in Q$ in the ball of radius εd centered at q'_i, where d is the minimum pairwise distance among points in Q'.

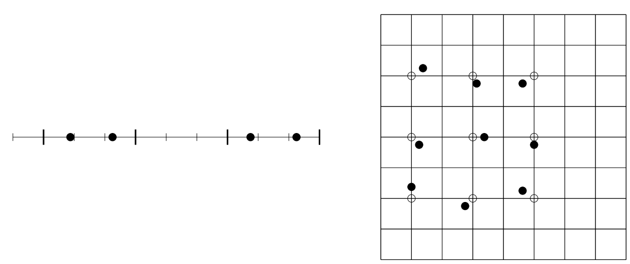

Figure 1: Left: a 4-term arithmetic progression (thick vertical bars) and a 1/3-approximate 4-term arithmetic progression (filled circles) on the line. Right: a 3-grid (empty circles) and a 1/4-approximate 3-grid (filled circles) in $\mathbb{R}^2$.

The condition $\varepsilon \leq 1/3$ is imposed to ensure that any two balls of radius εd around points in Q' are disjoint, and moreover, that any two distinct points of Q are separated by a constant times d, in this case by at least $d/3$.[2] In our theorems, ε-approximate means ε-approximate homothetic copy. We start with ε-approximate arithmetic progressions on the line by

[2]The choice of the constant $1/3$ in this definition is rather arbitrary. One could relax this inequality and require $\varepsilon < 1/2$ instead, however this would allow two points in Q be close to each other, which may defeat the intent.

proving the following analogue of Theorem 3 for points on the line:

Theorem 6 *For every positive integer k, $c, \delta > 0$, and $0 < \varepsilon \le 1/3$, there exists a positive number $Z_0 = Z_0(k, c, \delta, \varepsilon)$ with the following property: Let S be a δ-separated point set in an interval I of length $|I| = L$ with at least cL points, where $L \ge Z_0$. Then S contains a k-point subset that forms an ε-approximate arithmetic progression of k terms. Moreover, one can set $Z_0(k, c, \delta, \varepsilon) = 2\delta \cdot (ks)^j$,* where $s = \lceil \frac{1}{\varepsilon} \rceil$, $r = \frac{k}{k-1}$, $j = \left\lceil \frac{\log \frac{2}{c\delta}}{\log r} \right\rceil$.

The next proposition shows that the separation condition in the theorem is necessary, for otherwise, even a 3-term approximate arithmetic progression cannot be guaranteed, irrespective of the size of the point set.

Proposition 1 *For any n and $0 \le \varepsilon < 1/3$, there exists a set of n points in $[0, 1]$, without an ε-approximate arithmetic progression of 3 terms.*

Remark. The following slightly different form of Proposition 1 may be convenient: For any n there exists a set of n points in $[0, 1]$, without an ε-approximate arithmetic progression of 3 terms, for any $0 \le \varepsilon \le 1/4$. For the proof, take $S = \{1/8^i \mid i = 0, \ldots, n-1\}$, and proceed in the same way.

For a d-dimensional cube $\Pi_{i=1}^d [a_i, b_i]$, let us refer to $(a_1, \ldots, a_d)$ as the *first vertex* of the d-dimensional cube. We now continue with ε-approximate grids in $\mathbb{R}^d$ by proving the following analogue of Theorem 5 for points in $\mathbb{R}^d$:

Theorem 7 *For every positive integers d, k, and $c, \delta > 0$, and $0 < \varepsilon \le 1/3$, there exists a positive number $Z_0 = Z_0(d, k, c, \delta, \varepsilon)$ with the following property: Let S be a δ-separated point set in the d-dimensional cube $Q = [0, L]^d$, with at least cL^d points, where $L \ge Z_0$. Then S contains a subset that forms an ε-approximate k-grid in $\mathbb{R}^d$. Moreover, one can set $Z_0(d, k, c, \delta, \varepsilon) = 2\delta \cdot (ks)^j$,* where $s = \left\lceil \frac{\sqrt{d}}{\varepsilon} \right\rceil$, $r = \frac{k^d}{k^d - 1}$, $j = \left\lceil \frac{\log \frac{\kappa_d}{c\delta}}{\log r} \right\rceil$. *Here κ_d (in the expression of j) is a constant depending on d: $\kappa_d = \left\lceil \frac{3^d \cdot (d/2)!}{\pi^{d/2}} \right\rceil$*, if d is even, and $\kappa_d = \left\lceil \frac{3^d \cdot (1 \cdot 3 \cdots d)}{2 \cdot (2\pi)^{(d-1)/2}} \right\rceil$, if d is odd.

By selecting a sufficiently fine grid in Theorem 7, one obtains by similar means the following general statement for any pattern in $\mathbb{R}^d$:

Theorem 8 *For every positive integer d, finite pattern $P \subset \mathbb{R}^d$, $|P| = k$, and $c, \delta > 0$, and $0 < \varepsilon \le 1/3$, there exists a positive number $Z_0 = Z_0(d, P, c, \delta, \varepsilon)$ with the following property: Let S be a δ-separated point set in the d-dimensional cube $Q = [0, L]^d$, with at least cL^d points, where $L \ge Z_0$. Then S contains a subset that is an ε-approximate homothetic copy of P.*

The iterative procedures used in the proofs of Theorems 6, 7 and 8, yield very simple algorithms for computing the respective approximate homothetic copies given input point sets satisfying the imposed requirements. Each iteration takes linear time (in the number of points); see the full version for details. On the other hand, these requirements are currently too high, and it is likely that such copies exist under much weaker conditions.

Remark. The following connection between Theorem 6 and Szemerédi's Theorem 3 is worth making. If one makes abstraction of the bounds obtained, the qualitative statement in Theorem 6 can be obtained as a corollary from Theorem 3; see the full version for details. It is also worth noting that our proof of Theorem 6 is self contained and much simpler (from first principles) than the proof one gets from Szemerédi's theorem as described above. Moreover, the upper bound resulting from our proof is much better than that one gets from the integer theorem. That is, with the quantitative bounds included, the two theorems (6 and 7) cannot be derived as corollaries of the classical integer theorems. Indeed, as mentioned in the introduction no combinatorial proof is known for the higher dimensional generalization of Szemerédi's theorem due to Fürstenberg and Katznelson.

3 Almost collinear points

Let $0 < \varepsilon < 1$, and let S be a finite point set in $\mathbb{R}^d$. S is said to be ε-*collinear*, if in every triangle determined by S, two of its (interior) angles are at most ε. Note that in particular, this condition implies that an ε-collinear point set is contained in a section of a cylinder whose axis is a diameter pair of the point set, and with radius εD, where D is the diameter; the cylinder radius is at most $\frac{D}{2} \tan \varepsilon \le \varepsilon D$, for $\varepsilon < 1$.

Theorem 9 *For any dimension d, positive integer k, and $\varepsilon > 0$, there exists $N = N(d, k, \varepsilon)$, such that any point set S in $\mathbb{R}^d$ with at least N points has a subset of k points that is ε-collinear.*

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