Minimum Enclosing Area Triangle with a Fixed Angle

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Abstract

Given a set S of n points in the plane and a fixed angle $0 < \omega < \pi$, we show how to find all triangles of minimum area with angle ω that enclose S in $O(n \log n)$ time.

1 Introduction

In geometric optimization, the goal is often to find an optimal object or optimal placement of an object subject to a number of geometric constraints (see [AS98] for a survey). Examples include finding the smallest circle enclosing a point set or finding the placement of a unit circle that contains the maximum number of points from a given point set. In our setting, we study the following problem: given a set S of n points in the plane, we wish to find all of the triangles with minimum area and a fixed angle $0 < \omega < \pi$ that enclose S. When no constraint is put on the angles, Klee and Laskowski [KL85] gave an $O(n \log^2 n)$ time algorithm for finding the minimum area enclosing triangle. This was later improved to $O(n \log n)$ by O'Rourke et al. [OAMB86] which is optimal. Bose et al. [BMSS09] provided optimal algorithms for the setting where one wishes to find the minimum area isosceles triangles. The setting we explore here is in between the two. We place a restriction on the angle but do not insist on the triangle to be isosceles. Our solution, which we outline below, uses ideas from the solutions of Klee and Laskowski, O'Rourke et al. and Bose et al.

2 The ω-Cloud

Since the solution to the general problem only needs to consider the vertices of the convex hull of S, the first step is to compute the convex hull. In the remainder of the paper, we assume that the input is a convex n-gon with vertices given in clockwise order.

Let P be a convex n-gon. We denote the edges of P in clockwise order by e_i for $0 \leq i \leq n-1$ (all index manipulation is modulo n). Before presenting Step 1, we need two definitions.

Definition 1 (ω-wedge) *Let q be a point in the plane and ω be an angle $(0 < \omega < \pi)$. Let Δ and Δ' be two rays emanating from q such that the angle between Δ and Δ' is ω. We say that the closed set formed by q, Δ, Δ' and the points between Δ and Δ' create an ω-*wedge*. The point q is called the* apex *of the ω-wedge. An ω-wedge W* encloses *a polygon P when $P \subseteq W$ and both Δ and Δ' are tangent to P.*

Definition 2 (ω-cloud) *Let P be a convex n-gon and ω be an angle $(0 < \omega < \pi)$. By rotating an enclosing ω-wedge around P, the apex traces a sequence of circular arcs that we call an ω-*cloud *(refer to Figure 1).*

The idea behind the algorithm is to consider all possible ω-wedges that enclose P. Then for each of these ω-wedges, we find the minimal triangle by identifying a third side. We finally select the smallest ones among all possible such triangles.

Step 1: Compute the ω-cloud around P and denote it by Ω. Ω consists of $k = O(n)$ arcs of a circle [BMSS09] that we denote in clockwise order by Γ_i for $0 \leq i \leq k-1$. The meeting point of Γ_i and Γ_{i+1} is denoted by u_{i+1} for $0 \leq i \leq k-1$ (refer to Figure 1). We also refer to Γ_i by

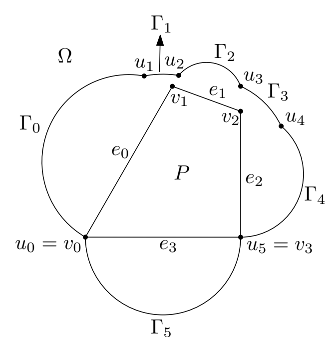

Figure 1: Step 1: Ω is the $\frac{1}{2}\pi$-cloud of P $(k = 6)$.

the closed set $[u_i, u_{i+1}]$ containing all the points of Γ_i. This step takes $O(n)$ time [BMSS09].

3 Optimal Solution Given a Fixed ω-Wedge

Step 2: Let $q \in \Omega$. Consider the ω-wedge defined by apex q and rays Δ and Δ' that encloses P. Find $b \in \Delta$ (respectively $c \in \Delta'$) such that P is enclosed in $\triangle qbc$ and the midpoint m of bc is on P

We claim that

1. it is always possible to find b and c satisfying these properties,

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2. $\triangle qbc$ is the minimum triangle enclosing P that can be constructed on this ω-wedge,

3. this step takes $O(n)$ time to compute.

Claims 1 and 2 where q is outside of P is proven by Klee and Laskowski in [KL85]. The proof of Claims 1 and 2 where q is on P is similar. A linear time algorithm (Claim 3) can be easily deduced from the proof of Claims 1 and 2.

4 Walking Around the ω-Cloud

In the previous step, we computed the minimum area triangle enclosing P that can be constructed on the ω-wedge of apex q and rays Δ and Δ'. The key idea is to consider all possible ω-wedges enclosing P and having its apex on Ω. For each of these ω-wedges, we can compute the minimum triangle enclosing P as we did in the previous step. The final solution is the minimum among these triangles. Of course, there are infinitely many such triangles. However, as we will show, we only need to consider $O(n)$ of them.

In what follows, we consider the points of Ω and the related ω-wedges in clockwise order. When thinking of all possible ω-wedges enclosing P and having its apex on Ω, one must pay attention to the following case. If q is the endpoint of one of the circular arcs of Ω and also one of the vertices of P, then there are infinitely many ω-wedges enclosing P and having q as an apex to consider (refer to Figure 2).

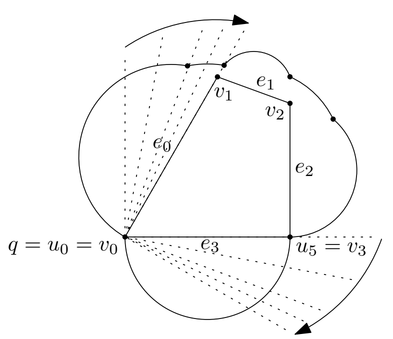

Figure 2: $\frac{1}{2}\pi$-wedge turning clockwise around a corner.

The following lemma describes the behaviour of m (the midpoint of bc constructed in Step 2) as q moves clockwise around Ω.

Lemma 1 *As an ω-wedge moves clockwise such that its apex q stays on Ω, the midpoint m of bc moves clockwise along P. Specifically, take $q', q'' \in \Omega$ with q'' clockwise from q'.*

1. *If $q' \in \Gamma_i \setminus \{u_{i+1}\}$ $(0 \le i \le k-1)$, $q'' \in]q', u_{i+1}]$, and q' and q'' are close enough to each other, then we have the following:*

   (a) *If $m' \in e_j \setminus \{v_j, v_{j+1}\}$ $(0 \le j \le n-1)$, then $m'' \in]m', v_{j+1}] \subset e_j$.*

   (b) *If $m' = v_j$ $(0 \le j \le n-1)$, then $m'' \in e_j$ (possibly $m'' = m'$).*

2. *If $q' = q'' = u_i$ $(0 \le i \le k-1)$ and $u_i = v_l$ $(0 \le l \le n-1)$. Let W' and W'' be two different ω-wedges enclosing P with $q' = q'' = u_i$ as an apex. If W'' is clockwise from W' (refer to Figure 2) and W' and W'' are close enough to each other, then we have the following:*

   (a) *If $m' \in e_j \setminus \{v_j, v_{j+1}\}$ $(0 \le j \le n-1)$, then $m'' \in]m', v_{j+1}] \subset e_j$.*

   (b) *If $m' = v_j$ $(0 \le j \le n-1)$, then $m'' \in e_j$ (possibly $m'' = m'$).*

Proof: Without loss of generality, $b'c'$ is on the abscissa (refer to Figure 3).

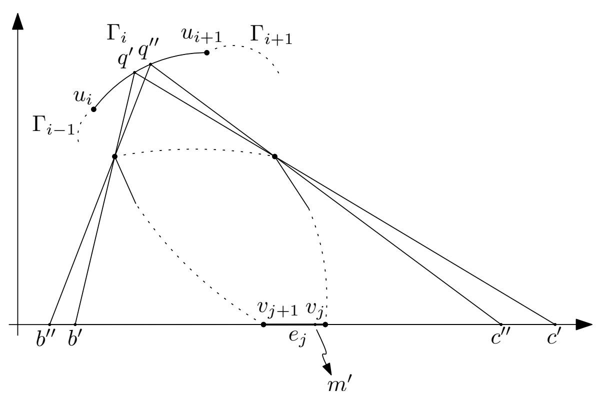

Figure 3: Proof of Lemma 1.

1. (a) If q' and q'' are close enough to each other, then $m'' \in e_j$. Therefore $b'' = (b''_x, 0)$, $b' = (b'_x, 0)$, $c'' = (c''_x, 0)$ and $c' = (c'_x, 0)$ with $b''_x < b'_x < c''_x < c'_x$. Hence $\frac{b''_x + c''_x}{2} < \frac{b'_x + c'_x}{2}$, so $m'' \in]m', v_{j+1}] \subset e_j$.

   (b) If q' and q'' are close enough to each other, then the only situation to discard is $m'' \in e_{j-1} \setminus \{v_j\}$. So suppose $m'' \in e_{j-1} \setminus \{v_j\}$ for a contradiction. Therefore m' and m'' both belong to e_{j-1} Hence an argument similar to the one of Point 1a leads to $m'' = v_j$, which is a contradiction.

2. The proof is similar to the one of Point 1. □

Step 3: Move the ω-wedge of apex q clockwise along Ω and maintain b, c and m as defined in the previous step. Collect all of the following three types of event points (refer to Figure 4).

1. $q = u_i$ for a $0 \le i \le k-1$.

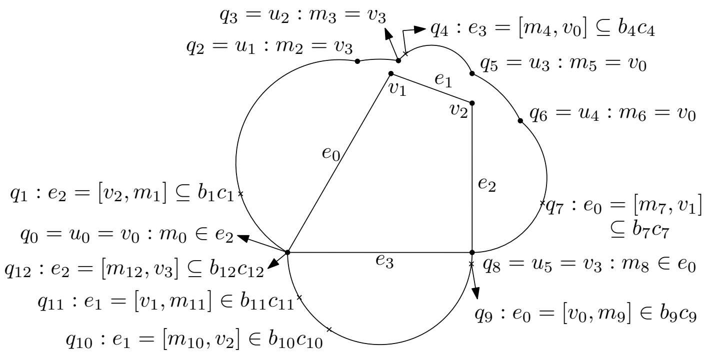

Figure 4: STEP 3: $q_0 = u_0 = v_0$ is such that $m_0 \in e_2$. Note that $u_0 = v_0$ gives birth to two different event points, namely q_0 and q_{12}. They correspond to the $\frac{1}{2}\pi$-wedge enclosing P and having $u_0 \in \Gamma_0$ as a vertex, and the $\frac{1}{2}\pi$-wedge enclosing P and having $u_0 \in \Gamma_5$ as a vertex. $q_5 = u_3 : m_5 = v_0$ means that $m_5 = v_0$ and $b_5c_5 \cap P = \{v_0\}$. At such a place, m does not move even though q does.

2. q is such that $e_i = [m, v_{i+1}] \subseteq bc$ for a $0 \leq i \leq n-1$.
3. q is such that $e_i = [v_i, m] \subseteq bc$ for a $0 \leq i \leq n-1$.

For each event point, save its type together with the location of the corresponding midpoint m.

There are $k = O(n)$ event points of the first type. Since there are n edges, then there are at most n event points of the second type and n event points of the third type. Therefore, this step takes $O(n)$.

5 Finding the Optimal Solution When the Apex is on a Circular Arc

Between any two consecutive event points, there is a single arc of a circle (that might be reduced to a single point if two consecutive event points are both on one of the vertices of P). Consider such an arc of a circle. As an ω-wedge moves clockwise such that its apex q stays on this circular arc, the minimum triangle enclosing P changes. For a fixed arc of a circle between two event points, what is the optimal triangle?

The point q is either on one of the vertices of P or on one of the circular arcs of Ω, and the midpoint m of bc either stays on one of the vertices of P or moves on one of the edges of P. Therefore, there are four possible cases to contend with. The two cases where q is on one of the vertices of P can easily be solved in constant time. The two cases where q is on one of the circular arcs of Ω need to be outlined in more detail.

The following lemma explains how to deal with the case where q is on one of the circular arcs of Ω and the midpoint m of bc stays on one of the vertices of P.

Lemma 2 *Let $\Gamma = [v, w]$ be one of the circular arcs of Ω and v_j be a point. It is possible to find the triangle $\triangle qbc$ of minimum area such that $q \in \Gamma$, q, v and b lie on the same line, q, w and c lie on the same line, and $v_j \in bc$ in $O(1)$ time.*

Proof: The strategy is to first fix an ω-wedge W with apex $q \in \Gamma$ and then use the proposition of Klee and Laskowski [KL85] (Claim 2) to construct the minimum triangle for W. Then we move W around Γ while maintaining the minimum triangle. It remains to look, among all these triangles, which has the smallest area.

Without loss of generality, Γ is the locus of points q such that $\angle vqw = \omega$. Hence we can take $v = (0, 0)$ and $w = (2r\sin(\omega), 0)$ where r is the radius of Γ (refer to Figure 5). Let $\alpha = \angle v_j vw$, $\beta = \angle v_j wv$ and $\theta = \angle qvw$

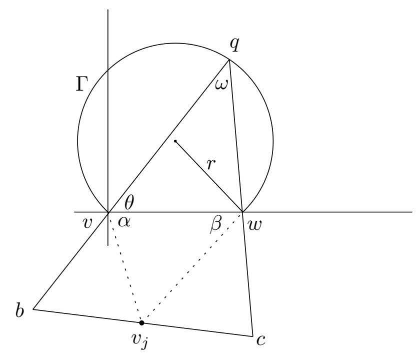

Figure 5: Proof of Lemma 2.

By geometry and trigonometry, we get that the area $\sigma(\theta)$ of $\triangle qbc$ is

$$-\frac{8r^2\sin(\alpha)\sin(\beta)\sin(\omega)}{\sin^2(\alpha+\beta)}\sin(\beta-\omega-\theta)\sin(\alpha+\theta) \ .$$

Then

$$\sigma'(\theta) = \frac{8r^2\sin(\alpha)\sin(\beta)\sin(\omega)}{\sin^2(\alpha+\beta)}\sin(\alpha-\beta+\omega+2\theta) \ .$$

To find the candidates for the minima of σ, we solve

$$\sin(\alpha-\beta+\omega+2\theta) = 0 \ .$$

Since the domain of θ is restricted, the only solution is $\theta_{opt} = \frac{\pi-\alpha+\beta-\omega}{2}$. Since $\sigma''(\theta_{opt}) < 0$, then the value of θ that minimizes σ is at one of the endpoints of the domain of θ. □

The following lemma explains how to deal with the case where q is on one of the circular arcs of Ω and the midpoint m of bc moves on one of the edges of P.

Lemma 3 *Let $\Gamma = [v, w]$ be one of the circular arcs of Ω and Δ'' be a line. It is possible to find the point $q \in \Gamma$ such that the line through qv, the line through qw and Δ'' form a triangle of minimum area in $O(1)$ time.*

Proof: Without loss of generality, Γ is the locus of points q such that $\angle vqw = \omega$. Hence we can take $v = (0, 0)$ and $w = (2r\sin(\omega), 0)$ where r is the radius of Γ (refer to Figure 6). We write the general equa-

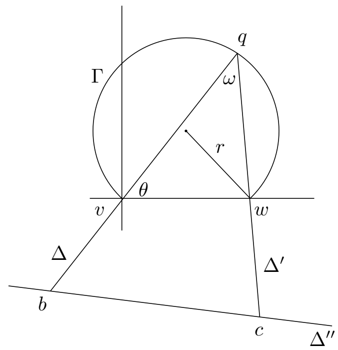

Figure 6: Proof of Lemma 3.

tion of Δ'' as $y = sx + t$ and we suppose that $s < 0$. Note $\theta = \angle qvw$. Also note Δ (respectively Δ') the line through q and v (respectively through q and w). Note b (respectively c) the meeting point of Δ and Δ'' (respectively of Δ' and Δ'').

By geometry and trigonometry, we get

$$\begin{aligned} q &= \left(2r\frac{\sin(\omega)X^2+\cos(\omega)X}{X^2+1}, 2r\frac{\sin(\omega)X+\cos(\omega)}{X^2+1}\right), \\ b &= \left(\frac{-tX}{sX-1}, \frac{-t}{sX-1}\right), \\ c &= \left(\frac{(2r\sin^2(\omega)+t\cos(\omega))X+2r\sin(\omega)\cos(\omega)-t\sin(\omega)}{(\sin(\omega)-s\cos(\omega))X+\cos(\omega)+s\sin(\omega)},\right. \\ &\quad \left.\frac{(2r\sin(\omega)s+t)(\sin(\omega)X+\cos(\omega))}{(\sin(\omega)-s\cos(\omega))X+\cos(\omega)+s\sin(\omega)}\right), \end{aligned}$$

where $X(\theta) = \cot(\theta)$. The area $\mu(X)$ of $\triangle qbc$ is

$$\frac{1}{2}\sin(\omega)\times \frac{((2sr\sin(\omega)+t)X^2+2r(s\cos(\omega)-\sin(\omega))X+t-2r\cos(\omega))^2}{(1-sX)(s\sin(\omega)+\cos(\omega)-(s\cos(\omega)-\sin(\omega))X)(1+X^2)}.$$

Then $\mu'(X) = \frac{p_2(X)p_4(X)}{p_8(X)}$ where p_i is a polynomial of degree i with coefficients depending on ω, r, s and t for $i = 2, 4, 8$.

Therefore $\mu'(X) = 0$ if and only if $p_2(X) = 0$ or $p_4(X) = 0$. Polynomials of degree $d \leq 4$ can be solved exactly in constant time [DF03]. Then it remains to solve for θ. Therefore we get six candidates from p_2 and p_4 for the minimum of $\mu(\theta)$. There are two more candidates which correspond to the endpoints of the domain of θ. Finally, $\mu'(X(\theta)) = \mu'(X)X'(\theta) = \mu'(X)\csc(\theta)\cot(\theta)$, so $\theta = \frac{1}{2}\pi$ is also a candidate.

If $s \geq 0$ or Δ'' is vertical, the proof is similar. □

Step 4: For each pair of consecutive event points q_i and q_{i+1}, compute the minimum triangle enclosing P and having a vertex on the circular arc $[q_i, q_{i+1}]$.

6 Putting it All Together

Step 5: Take the minimum triangles among the ones computed in the previous step.

Therefore this step takes $O(n)$ time to compute and the algorithm also takes $O(n)$ time to compute. For a general set of points, first computing the convex hull leads to $O(n \log n)$ time of computation.

7 A Note on the Complexity of the Solution

In this section, we show that the model of computation needed to solve the general problem exactly must include square root and cubic root. One of the cases in Section 5 required us to find the roots of a degree 4 polynomial, which in general implies using square roots and cubic roots. We show that it cannot be avoided in this problem by providing an example of a set of points where the optimal solution lies on the root of an irreducible degree 4 polynomial.

Consider the quadrilateral $Q = [abcd]$ where $a = (0,0)$, $b = (2,0)$, $c = \left(2, -\frac{3}{2}\right)$ and $d = \left(-4\frac{4\sqrt{3}-1}{47}, 4\frac{\sqrt{3}-12}{47}\right)$. The right triangle of minimum area that encloses Q is constructed such that the hypotenuse is on cd. We need to solve the quartic equation of Lemma 3 to compute this solution exactly. The precise expression for $p_4(X)$ in this setting given Q as an input is $p_4(X) = 13X^4 - 92X^3 + 45X^2 + 12X - 62$. Since $p_4(X)$ is irreducible over $\mathbb{Q}[X]$ and $disc(p_4) = -2^5 \cdot 17^6 \cdot 5689$ is not a perfect square, the algebraic expressions for the roots of p_4 must be written with square roots and cubic roots. Moreover, they cannot be simplified (see [DF03]).

Therefore, in general, we cannot avoid square roots nor cubic roots in the computation of the minimum enclosing area triangle with a fixed angle.

8 Conclusion

We have shown how to compute all triangles of minimum area with a fixed angle $0 < \omega < \pi$ that enclose a point set. It would be interesting to see if some of these techniques generalize to other settings or other optimization criteria. For example, minimizing the perimeter of a triangle enclosing a point set or finding the smallest area tetrahedron of a set of points in 3-space.

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