Approximating the Independent Domatic Partition Problem in Random Geometric Graphs – An Experimental Study

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Abstract

We investigate experimentally the Domatic Partition (DP) problem, the Independent Domatic Partition (IDP) problem and the Idomatic partition problem in Random Geometric Graphs (RGGs). In particular, we model these problems as Integer Linear Programs (ILPs), solve them optimally, and show on a large set of samples that RGGs are independent domatically full most likely (over 93% of the cases) and domatically full almost certainly (100% of the cases). We empirically confirm using two methods that RGGs are not idomatic on any of the samples. We compare the results of the ILP-based exact algorithms with those of known coloring algorithms both centralized and distributed. Coloring algorithms achieve a competitive performance ratio in solving the IDP problem [11, 10]. Our results on the high likelihood of the "independent domatic fullness" of RGGs lead us to believe that coloring algorithms can be specifically enhanced to achieve a better performance ratio on the IDP size than [11, 10]. We also investigate experimentally the extremal sizes of individual dominating and independent sets of the partitions.

1 Introduction and Motivation

The domatic partition (DP) problem is a classical problem in graph theory whose goal is to partition a graph G into disjoint dominating sets. The *domatic number* $d(G)$ is the maximum number of dominating sets in such a partition [3]. The concept has an important application to energy conservation and sleep scheduling in Wireless Sensor Networks (WSN) [16, 15, 7, 9, 11, 10] which are often modeled in practice as Random Geometric graphs (RGGs). A random geometric graph $G(n, r)$ is defined by n vertices uniform in the unit square with an edge between any two vertices of V within Euclidean distance r of each other. An RGG simply induces a uniform probability distribution on a Unit Disk Graph (UDG). A variation of the DP problem is the Independent Domatic Partition (IDP) problem which seeks to partition a graph G into disjoint independent dominating sets. The *independent domatic number* $d_{ind}(G)$ is the maximum size of such a partition.

For any graph G, $d_{ind}(G) \leq d(G) \leq \delta(G) + 1$ where $\delta(G)$ denotes the minimum degree in G. If $d(G) = \delta(G)+1$ and/or $d_{ind}(G) = \delta(G)+1$, then G is called *domatically full* (DF) and/or *independent domatically full* (IDF) respectively [3]. A graph whose vertices V can be strictly partitioned into disjoint independent dominating sets is termed *indominable* [1] or *idomatic* [3]. The *idomatic number* $id(G)$ is the partition's maximum size. Notice that the DP, IDP and idomatic problems are all NP-complete in general graphs [2, 4, 8] and also believed to be so in UDGs [13].

The study described herein is motivated by the desire to empirically verify the existence of the upper bound of $\delta + 1$ disjoint independent dominating sets in RGGs (which model Wireless Sensor Networks). Namely, are random geometric graphs *independent domatically full* in practice?

Moreover, we experimentally study the "domination chain" $\gamma(G) \leq i(G) \leq \beta_0(G)$ in RGGs. The "domination chain" is a relation between graph parameters that is satisfied in any graph G [3], where $\gamma(G)$ is the size of the minimum dominating set (MDS) termed the *domination number*, $i(G)$ is the size of the minimum independent dominating set (MIDS) termed the *independence domination number* and $\beta_0(G)$ is the size of the maximum independent set (MaxIS) termed the *independence number*. Finding these values are NP-complete problems in general graphs and Unit Disk Graphs [3, 12].

2 Our Contributions

In this paper, our main contributions are:

-We solve the IDP problem optimally and show that over 93% of the RGG instances are independent domatically full and 100% of the instances are domatically full. The high likelihood of the existence of an optimal partition of $\delta + 1$ independent dominating sets in typical RGGs suggests that coloring algorithms can be fine-tuned to achieve a better performance ratio [11, 10].

-We confirm by Smallest Last (SL) coloring [14] for a large sample of RGG instances that $\chi(G) \geq \omega(G) > \delta(G) + 1$, hence these graphs cannot be *idomatic* [1]. In addition, we formulate the idomatic partition prob-

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lem as an ILP and confirm through experiments that all graphs of the sample are not *idomatic*.

-We experimentally study the node packing in the sets of the IDP solution and also report on the domination chain values and compare the results obtained by ILP algorithms and coloring algorithms with the asymptotic bounds based on "optimal" triangular lattice packing.

We believe this study answers relevant questions for practitioners and also stimulates further research on the approximability of the IDP problem in UDGs and RGGs and on the asymptotic behavior of domination and domatic properties in RGGs.

3 Algorithms

IDP Formulation. Given a graph $G = (V, E)$ and the set $K = \{1, ..., \delta+1\}$, we formulate the IDP problem as the following Integer Linear Program (ILP):

$$\text{maximize} \sum_{k=1}^{\delta+1} u_k$$

$$\text{s.t. } x_u^k + \sum_{v:(u,v)\in E} x_v^k \geq u_k \ \forall u \in V, k \in K \quad (1)$$

$$x_v^k + x_u^k \leq 1 \ \forall u, v \in V : (u, v) \in E, k \in K \quad (2)$$

$$\sum_{k=1}^{\delta+1} x_u^k \leq 1 \ \forall u \in V \quad (3)$$

$$u_k \in \{0, 1\},\ x_u^k \in \{0, 1\} \ \forall u \in V, k \in K \quad (4)$$

where u_k=1 if dominating set $S_k = \{u|x_u^k = 1\}$ is selected in the IDP and u_k=0 otherwise. Constraint (1) expresses domination, (2) independence, (3) node disjointness, i.e. a node can be part of at most one set, and (4) variable integrality. We also formulate the idomatic partition problem as an ILP where we maximize the size of the independent domatic partition as well as the total number of packed nodes in the sets of the partition. The exact algorithms for the MDS, MIDS and MaxIS problems are also modeled as ILPs.

Coloring Heuristics. We use 5 centralized graph coloring heuristics [11] and 4 distributed ones [10] to experimentally approximate the IDP problem.

4 Experimental Results

In this paper, ILP models are solved optimally using CPLEX 10.0 installed on a Dual Quad Core Intel Xeon X5570 with 72 GB RAM running CentOS Linux 2.6.18. Each core is clocked at 3.00 GHz. The coloring algorithms are implemented in *C#.Net* (Microsoft Visual Studio 2005) on an Intel Core 2 Duo E8400 processor clocked at 3.00 GHz with 3 GB RAM running Windows Vista Enterprise SP1. Our data set consists of 15 graphs generated randomly with $\delta \in$ {5,10,20} and $n \in$ {50,100,200,400,800}. Results for each (δ, n) pair are averaged over 20 RGG instances, except for the $(\delta = 20, n = 800)$ case where we average over 10 instances, given that the running times of the ILP models were prohibitively long. This provides a sample of 290 test RGG instances that we choose all to be connected. An RGG instance of parameters (δ, n) is selected as follows: First, we generate all n vertices' (x,y) coordinates i.u.d in the unit square then we sort in non-decreasing order all possible $n(n - 1)/2$ edges by their Euclidean distance. Following an evolutionary random graph generation paradigm [5], we add the edges to the graph one-by-one in increasing length until the minimum degree over all n vertices equals δ. The edge length that achieves the desired δ represents r of the graph $G(n, r)$. The values of δ are picked to be representative of WSNs modeled as RGGs where typical node degrees cannot be too high. The exact ILP-based algorithms have a running time that can be exponential in the size of the input, whereas the coloring heuristics achieve a competitive performance ratio on RGGs in polynomial time.

4.1 Domination and Independence in RGGs

Table 1 reports the exact values of the domination chain parameters $\gamma(G)$, $i(G)$ and $\beta_0(G)$ by solving the ILP models of the MDS, MIDS and MaxIS problems. For indicative purposes, we report the average radius $\overline{r}$ calculated over the set of 20 r values selected to achieve the desired δ for each one of the 20 RGG instances representative of a given (δ, n) pair. Based on a triangular lattice packing argument, we showed in [9, 11] lower and upper bounds on the size of a maximal (dominating) independent set, which we denote respectively as i^{tr} and β_0^{tr}. Namely, $i^{tr} = \frac{1}{3}.[1/(r^2\frac{\sqrt{3}}{2})]$ and $\beta_0^{tr} = 1/(r^2\frac{\sqrt{3}}{2})$. We also use $\beta_n(r) = (1 + 1/r)^2$ as the absolute upper bound on the size of a maximum independent set in a random geometric graph $G(n, r)$ [9].

We observe that $i^{tr} \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \beta_0^{tr} \leq \beta_n(r)$. However, in certain cases, e.g. $(\delta = 10, n = 50)$, we have $\beta_0(G) > \beta_0^{tr}$. In other words, the computed exact value of the *independence number* $\beta_0(G)$ is greater than the expected triangular lattice-based upper bound β_0^{tr}. We attribute this to a boundary effect in the unit square which produces a value of β_0^{tr} smaller than if we had an infinite unbounded lattice. Furthermore, we report that in 288 cases out of 290 (99.3%), $\gamma(G) = i(G)$. By Theorem [3], if G is a graph containing no induced subgraph isomorphic to $K_{1,3}$ (i.e. G is claw-free), then $\gamma(G) = i(G)$. We verified, however, that all graphs have, in fact, at least one claw. This is simply an empirical verification that the theorem is a conditional but not a biconditional.

Table 2 shows the extremal sizes of individual independent dominating sets obtained in the IDP partitions. For lack of space, we only show the results of the ILP exact model and those of two greedy coloring heuristics: Smallest Last (SL), a centralized topology-based algorithm that first orders the vertices recursively by deleting minimum degree vertices, and then assigns colors in the reverse "smallest last" order [14]; and Distributed Lexicographic (DLX), a distributed geometry-aware algorithm that assigns colors distributively respecting the order of the vertices' x coordinates (with ties broken according to the y coordinates) [10]. i_{ILP}, β_{ILP}, i_{SL}, β_{SL}, i_{DLX} and β_{DLX} represent the minimum and maximum size among all independent dominating sets obtained in the IDP partition solution by the ILP model, SL and DLX coloring methodology respectively. In Table 2, we observe that coloring heuristics pack more vertices in any single set than ILP, i.e. β_{SL} and β_{DLX} are closer to the upper bounds $\beta_0(G)$ or β_0^{tr} than β_{ILP} is.

Table 1: Domination chain values.

$\delta, n, \overline{r}$	i^{tr}	$\gamma(G)$	$i(G)$	$\beta_0(G)$	β_0^{tr}	$\beta_n(r)$
5,50,0.41	2.40	3.65	3.65	7.20	7.20	12.1
5,100,0.30	4.70	6.10	6.10	12.6	14.3	20.3
5,200,0.19	10.8	12.1	12.2	26.2	32.4	39.6
5,400,0.14	18.4	19.9	19.9	44.5	55.3	62.7
5,800,0.10	40.4	40.4	40.4	91.0	121	126
10,50,0.52	1.40	2.50	2.55	4.90	4.28	8.50
10,100,0.38	2.60	4.00	4.00	8.65	7.95	13.1
10,200,0.26	5.70	7.15	7.15	16.4	17.2	23.5
10,400,0.19	10.5	12.6	12.6	29.3	31.8	38.9
10,800,0.13	23.2	25.2	25.2	59.9	69.8	76.9
20,50,0.72	0.70	1.15	1.15	3.90	2.20	5.70
20,100,0.51	1.40	2.80	2.80	5.50	4.30	8.67
20,200,0.37	2.80	4.10	4.10	9.95	8.50	13.8
20,400,0.26	5.50	7.30	7.30	17.4	16.7	23.0
20,800,0.18	11.4	14.3	14.3	35.7	34.3	41.6

Table 2: Min/Max independent dominating sets sizes.

δ, n	i_{ILP}	β_{ILP}	i_{SL}	β_{SL}	i_{DLX}	β_{DLX}
5,50	3.90	5.10	4.55	6.10	4.40	6.45
5,100	7.20	9.05	9.50	10.9	8.85	11.8
5,200	14.5	17.7	19.7	22.0	20.5	23.4
5,400	23.1	26.5	34.7	37.2	35.7	40.4
5,800	47.1	52.4	74.8	77.3	77.3	82.5
10,50	2.60	3.90	2.80	4.35	2.80	4.65
10,100	4.20	6.10	5.45	7.40	5.40	8.05
10,200	7.90	10.2	11.6	13.7	10.8	14.8
10,400	13.6	16.6	21.4	24.5	21.2	26.5
10,800	28.7	33.1	45.9	49.6	47.0	52.6
20,50	1.15	3.00	1.15	2.95	1.15	3.25
20,100	2.85	3.95	3.10	4.85	3.15	5.35
20,200	4.15	6.30	5.85	8.45	5.85	9.30
20,400	7.65	10.2	11.7	14.9	11.5	16.2
20,800	15.7	20.4	23.1	27.5	22.4	29.3

Table 3: Non independent domatically full instances.

$(5,100)$	$(5,800)$	$(10,50)$	$(20,50)$	$(20,100)$
95%(1)	90%(1)	95%(1)	40%(1,6)	85%(1)

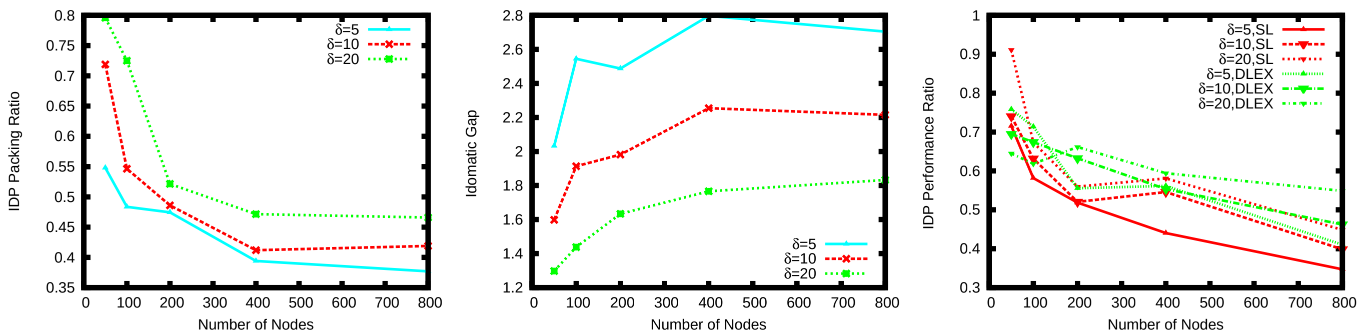
4.2 Independent Domatic Partitions in RGGs

We report that all 290 experimented RGG instances were domatically full and 271 (over 93%) were independent domatically full (IDF). Namely, the cases $(5,50)$, $(5,200)$, $(5,400)$, $(10,100)$, $(10,200)$, $(10,400)$, $(10,800)$, $(20,200)$, $(20,400)$ and $(20,800)$ were all IDF. Table 3 shows the (δ, n) pairs where some instances are not IDF. For each (δ, n) pair, we report the percentage of random instances that are IDF, the second value(s) between parentheses denotes the number of sets (or min and max number of sets) that are missed compared to the upper bound $(\delta + 1)$. For example, in the $(\delta = 20, n = 50)$ case, 40% of the 20 instances were IDF, the lowest gap from $\delta + 1$ is one set, and the highest is 6 sets. The pattern we observe is that when δ is very close to n (a highly dense graph), the graph has a higher chance not to be independent domatically full.

We define the IDP packing ratio as the portion of nodes of V in the $d_{ind}(G)$ independent dominating sets. Figure 1a shows the evolution of the ratio as n grows for various δ. For a fixed δ, the ratio decreases with increasing n, and it increases for fixed n as δ increases. We derive from [1] that if $\chi(G) \geq \omega(G) > \delta(G)+1$ then G is not idomatic. We use $\omega_{SL}(G)$ as a lower bound on the clique number obtained by Smallest Last coloring [11] and report that in all samples, $\omega_{SL}(G) > \delta(G) + 1$, therefore the graphs are not idomatic. We also confirm this observation by solving the ILP model of the Idomatic partition problem. We define the Idomatic gap as the ratio of the maximal clique value ω_{SL} over $\delta + 1$ and conjecture that the closer the ratio is to 1, the more likely the graph is to be idomatic. We observe that the Idomatic gap is correlated with the IDP packing ratio. Intuitively, the larger the Idomatic gap is, the lower is the IDP packing ratio. Figure 1b shows the evolution of the Idomatic gap as n grows for various δ. Figure 1c shows the performance ratio on $d_{ind}(G)$ obtained by SL and DLX. We observe that the ratio decreases as n increases and it is generally higher for the same n when δ increases. Notice that these ratios are obtained as a by-product of the coloring algorithms whose purpose is unrelated to approximation of the IDP problem.

5 Conclusion and Future Work

We have shown experimentally that RGGs are domatically full in all instances and independent domatically full in 93% of the instances. Strongly chordal (SC) graphs are provably domatically full [2, 6]. Further re-



(a) IDP Packing Ratio (b) Maximal Clique to $\delta + 1$ ratio (c) IDP Perf. Ratio for various δ

Figure 1: Performance of the Independent Domatic and Idomatic Partitions for various δ.

search related to this work includes the problem of determining whether the experimented graphs are strongly chordal which would explain their domatic fullness. A more general question is are RGGs strongly chordal with high likelihood? Another direction we are pursuing is how do we enhance the coloring algorithms to improve their performance ratio in solving the IDP problem.

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