On Polygons Excluding Point Sets

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Abstract

By a polygonization of a finite point set S in the plane we understand a simple polygon having S as the set of its vertices. Let B and R be sets of blue and red points, respectively, in the plane such that $B \cup R$ is in general position, and the convex hull of B contains k interior blue points and l interior red points. Hurtado et al. found sufficient conditions for the existence of a blue polygonization that encloses all red points. We consider the dual question of the existence of a blue polygonization that excludes all red points R. We show that there is a minimal number $K = K(l)$, which is a polynomial in l, such that one can always find a blue polygonization excluding all red points, whenever $k \geq K$. Some other related problems are also considered.

1 Introduction

Let S be a set of points in the plane in *general position*, i.e., such that no three points in S are collinear. A *polygonization* of S is a simple (i.e., closed and non-self-intersecting) polygon P such that its vertex set is S. Polygonizations of point sets have been studied a lot recently (e.g. [6, 3, 1]).

We say that a polygon P *encloses* a point set V if all the points of V belong to the interior of P. If all the points of V belong to the exterior of P, then we say that P *excludes* V. Let B and R be disjoint point sets in the plane such that $B \cup R$ is in general position. The elements of B and R will be called *blue* and *red* points, respectively. Also, a polygon whose vertices are blue is a *blue polygon*. A polygonization of B is called a *blue polygonization*. Throughout the paper in the figures we depict a blue point by a black disc, and a red point by a black circle.

Let $conv(X)$ denote the convex hull of a subset $X \subseteq \mathbb{R}^2$. By a *vertex* of $conv(X)$ we understand a 0-dimensional face on its boundary. We assume that all the red points belong to the interior of $conv(B)$, since we can disregard red points lying outside $conv(B)$ for the problems we consider. Let $n \geq 3$ denote the number of vertices of $conv(B)$, $k \geq 1$ the number of blue points in the interior of $conv(B)$, and $l \geq 1$ the number of red points (which all lie in the interior of $conv(B)$ by our assumption).

In [2, 5] the problem of finding a blue polygonization that encloses the set R was studied, and in [5] Hurtado et al. showed that if the number of vertices of $conv(B)$ is bigger than the number of red points, then there is a blue polygonization enclosing the set R. Moreover, they showed by a simple construction that this result cannot be improved in general.

We propose to study a dual problem, where the goal is to find conditions under which there is a blue polygonization excluding the red points (Figure 1).

Our main result is the following theorem.

Theorem 1 *Let B and R be blue and red point sets in the plane such that $B \cup R$ is in general position and R is contained in the interior of $conv(B)$. Suppose l is the number of red points and k the number of blue points in the interior of $conv(B)$. Then there exists $k_0 = k_0(l) = O(l^4)$, so that whenever $k \geq k_0$, there exists a blue polygonization excluding the set R.*

Note that it is not a priori evident that such k_0 exists. We denote by $K(l)$ the minimum possible value $k_0(l)$ for which the above theorem holds. We also show that k_0 in Theorem 1 must be at least $2l - 1$.

Theorem 2 *For arbitrary $n \geq 3, l \geq 1$ and $k \leq 2l - 2$ there is a set of points $B \cup R$ (as before $|B| = n + k$, $|R| = l$ and the set of vertices of the convex hull of $B \cup R$ consists of n blue points) for which there is no polygonization of the blue points that excludes all the red points.*

We consider also a version of the problem where the goal is to use as few inner blue points as possible so as to form a blue polygon excluding the red set (Figure 2). We obtain the following result.

Theorem 3 *If $|B| = n + k$, $|R| = l$, $k \geq n^3 l^2$ and the convex hull of B contains k blue vertices in its interior, then there exists a simple blue polygonization of a subset of B of size at most $2n$ that contains all the vertices of the convex hull of B, and excludes all the red points.*

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Finally, we treat the following closely related problem. Given n red and n blue points in general position, we want to draw a polygon separating the two sets, with minimal number of sides. Our result is:

Theorem 4 *Let B and R be sets of n blue and n red points in the plane in general position. Then there exists a simple polygon with at most $3\lceil n/2 \rceil$ sides that separates blue and red points.*

Also, for every n there are sets B and R that cannot be separated by a polygon with less than n sides.

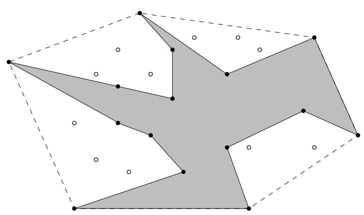


Figure 1: A blue polygonization excluding all the red points

2 Preliminary results

In this section we present several lemmas that we will use throughout the paper. Let us recall that B and R denote sets of blue and red points in the plane. We will assume that they are in general position, i.e., the set $B \cup R$ does not contain three collinear points. We will need the following useful lemma by García and Tejel [4].

Lemma 5 (Partition lemma) *Let P be a set of points in general position in the plane and assume that $p_1, p_2, \ldots, p_n$ are the vertices of the $conv(P)$ and that there are m interior points. Let $m = m_1 + \cdots + m_n$, where the m_i are nonnegative integers. Then the convex hull of P can be partitioned into n convex polygons $Q_1, \ldots, Q_n$ such that Q_i contains exactly m_i interior points (w.r.t. $conv(P)$) and $p_i p_{i+1}$ is an edge of Q_i. (Some interior points can occur on sides of the polygons $Q_1, \ldots, Q_n$ and for those points we decide which region they are assigned to.)*

The next corollary will be used as the main ingredient in the proof of Theorem 3.

Corollary 6 *If $|B| = |R| = n$ and the blue points are vertices of a convex n-gon, while all the red points are in the interior of that n-gon, then there exists a simple alternating $2n$-gon, i.e., a $2n$-gon in which any two consecutive vertices have different colors.*

In the proof of Theorem 1 we will be making a polygon by concatenating several polygonal paths obtained by the following proposition, which is rather easy (and whose proof we skip).

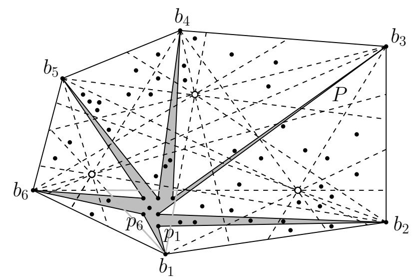


Figure 2: Alternating polygon using few inner blue vertices

Proposition 7 *Let S be a set of n points in the plane in general position and p and q two points from S. Then one can find a simple polygonal path whose endpoints are p and q and whose vertices are the n given points.*

In order to obtain by our method a bound on $K(l)$ ($|R| = l$) we need to take care of the situation, when the convex hull $conv(B)$ contains too many vertices. For that sake we have the following proposition, which can be established quite easily.

Proposition 8 *There exists a subset B' of B of size at most $2l + 1$, containing only the vertices of $conv(B)$, so that all the red points are contained in $conv(B')$.*

3 Proof of the main result

The aim of this section is to prove the main result, which is stated in Theorem 1, about sufficient conditions for the existence of a blue polygonization that excludes all the red points.

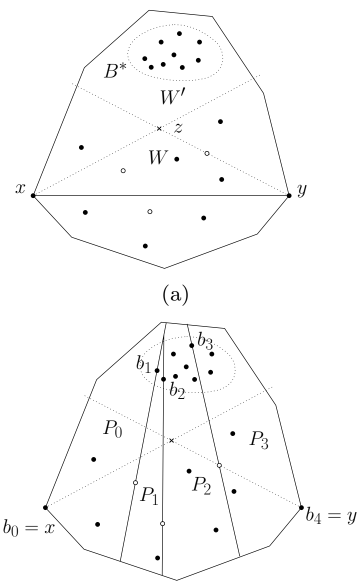
By a *wedge* with z as its apex point we mean a convex hull of two non-collinear rays emanating from z. We define an (l-)*zoo* $\mathcal{Z} = (B, R, x, y, z)$ (Figure 3(a)) as a set $B = B(\mathcal{Z})$ of blue and $R = R(\mathcal{Z})$, $|R| = l$, red points with two special blue points $x = x(\mathcal{Z}) \in B, y = y(\mathcal{Z}) \in B$ and a special point $z = z(\mathcal{Z})$ (not necessarily in B or R) such that:

1. every red point is inside $conv(B)$
2. x, y are on the boundary of $conv(B)$
3. every red point is contained in the wedge $W = W(\mathcal{Z})$ with apex z and boundary rays zx and zy.

We denote by $B^* = B^*(\mathcal{Z})$ the blue points inside $W' = W'(\mathcal{Z})$, the wedge opposite to $W(\mathcal{Z})$ (i.e., W' is the wedge centrally symmetric to W with respect to its apex). We refer to the points in B^* as to special blue points. We imagine x and y being on the x-axis (with x having smaller x-coordinate than y) and z being above it (see Figure 3(a)), and we are assuming that when we talk about objects being below each other in a zoo.

A *nice partition* of an l-zoo is a partition of $conv(B)$ into closed convex parts $P_0, P_1, \ldots, P_m$, for which there exist pairwise distinct special blue points $b_1, \ldots, b_m \in B^*$ (we call $b_0 = x$ and $b_{m+1} = y$) such that for every P_i we have that (see Figure 3(b)):

1. no red point is inside P_i, i.e., red points are on the boundaries of the parts
2. P_i has b_i and b_{i+1} on its boundary



(a)

(b)

Figure 3: (a) 3-zoo, (b) Nice partition of 3-zoo into 4 parts

A short proof of the next proposition is omitted.

Proposition 9 *Given a zoo $\mathcal{Z}$ with a nice partition, we can draw a polygonal path using all points of $B = B(\mathcal{Z})$ with endpoints $x(\mathcal{Z})$ and $y(\mathcal{Z})$ s.t. all the red points are below the polygonal path.*

The proofs of the following two lemmas can be found in the complete version of the article in the Electronic Proceedings.

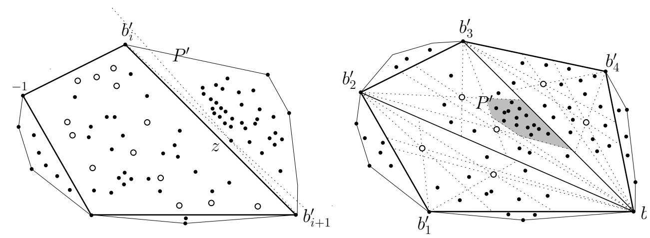
Lemma 10 *Given an l-zoo $\mathcal{Z}$, if $B^* = B^*(\mathcal{Z})$ contains a blue y-monotone convex chain of size $2l - 1$, then it has a nice partition.*

The next lemma is a variant of the previous one, and it is the key ingredient in the proof of the main theorem in this section.

Lemma 11 *Given an l-zoo $\mathcal{Z}$, if $B^* = B^*(\mathcal{Z})$ contains at least $\Omega(l^2)$ blue points, then it has a nice partition.*

Having the previous lemma, we are in the position to prove Theorem 1.

Proof. [Proof of Theorem 1.] First, by Proposition 8 we obtain a subset B', $|B'| = m$, of the vertices of $conv(B)$ of size at most $2l + 1$, so that $R \subseteq conv(B')$. Let $b'_0, b'_1, \ldots, b'_{m-1}$ denote the blue points in B' listed according to their cyclic order on the boundary of $conv(B')$. We distinguish two cases.



(a) **(b)**

Figure 4: Partition of $conv(B)$

1° *$conv(B')$ does not contain $\Omega(l^4)$ points in its interior.* It follows, that there is a convex region P' containing $\Omega(l^3)$ blue points, which is an intersection of $conv(B)$ with a closed half-plane T defined by a line through two consecutive vertices b'_i and b'_{i+1}, for some $0 \leq i < m$ (indices are taken modulo m), on the boundary of $conv(B')$, such that T does not contain the interior of $conv(B')$ (see Figure 4 (a)). Let B'' denote the set of vertices of $conv(B')$ except b'_i and b'_{i+1}. Observe that we have an l-zoo $\mathcal{Z}$ having $B(\mathcal{Z}) = B \setminus B''$, $R(\mathcal{Z}) = R$, b'_i and b'_{i+1} as $x(\mathcal{Z})$ and $y(\mathcal{Z})$, respectively. By the general position of B we can take $z(\mathcal{Z})$ to be a point very close to the line segment $b'_i b'_{i+1}$, so that $B^*(\mathcal{Z})$ contains $\Omega(l^2)$ blue points. Thus, by Lemma 11 we obtain a nice partition of Z. Hence, by Proposition 9 we obtain a blue polygonal path Q having $B \setminus B''$ as a set of vertices. The desired polygonal path is obtained by concatenating the path Q with the convex chain formed by the points in $B'' \cup \{b'_i, b'_{i+1}\}$.

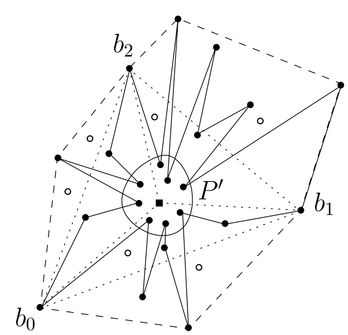


Figure 5: Forming a polygonization

2° *$conv(B')$ contains $\Omega(l^4)$ points in its interior.* Let R_i denote the intersection of R with the triangle $b'_0 b'_i b'_{i+1}$, for all $1 \leq i < m - 1$. For each triangle $b'_0 b'_i b'_{i+1}$ we consider the lines through all the pairs r and b, such that $b = b'_0, b'_i$ or b'_{i+1} and $r \in R_i$. For each i, $1 \leq i < m-1$, these lines partition the triangle $b'_0 b'_i b'_{i+1}$ into $O(|R_i|^2)$ 2-dimensional regions. Hence, by doing such a partition in all the triangles $b'_0 b'_i b'_{i+1}$ we partition $conv(B')$ into $O(\sum_{i=1}^{m-2} |R_i|^2) = O(|R|^2)$ regions, each of them fully contained in one of the triangles $b'_0 b'_i b'_{i+1}$. It follows that one of these regions, let us denote it by P', contains at least $\Omega(l^2)$ blue points (see Figure 4 (b)). Clearly, P' is contained in a triangle $b'_0 b'_i b'_{i+1}$, for some $1 \leq i < m-1$.

For the convenience we rename the points b'_0, b'_i, b'_{i+1} by b_0, b_1, b_2 in clockwise order. We apply Partition Lemma (Lemma 5) on the triangle $b_0 b_1 b_2$, so that we obtain a partition of the triangle $b_0 b_1 b_2$ into three convex polygonal regions P'_0, P'_1, P'_2 (in fact triangles), such that each part contains $\Omega(l^2)$ blue points belonging to $P' \cap P'_j$, for all $0 \leq j \leq 2$, and has $b_j b_{j+1}$ as a boundary segment. We denote by P_0, P_1, P_2 the parts in the partition of $conv(B)$, which is naturally obtained as the extension of the partition of $b_0 b_1 b_2$, so that P_j, $P_j \supseteq P'_j$, has $b_j b_{j+1}$ (indices are taken modulo 3) either as a boundary edge or as a diagonal.

In what follows we show that in each P_j, $0 \leq j \leq 2$, we have an l_j-zoo $\mathcal{Z}_j$, $l_j \leq l$, with b_j as $x(\mathcal{Z}_j)$ and b_{j+1} and $y(\mathcal{Z}_j)$, respectively, and with $\Omega(l^2)$ blue points in $B^*(\mathcal{Z}_j)$.

First, we suppose that there exists a red point in P'_j. We take $z(\mathcal{Z}_j)$ to be the intersection of two tangents t_1 and t_2 from b_j and b_{j+1}, respectively, to $conv(R \cap P'_j)$ that have $conv(R \cap P'_j)$ and $b_j b_{j+1}$ on the same side. Clearly, P' has to be contained in one of four wedges defined by t_1 and t_2. However, if P' is not contained in the wedge defined by t_1 and t_2, which has the empty intersection with the line through b_j and b_{j+1}, either P_{j+1} or P_{j-1} cannot have a non-empty intersection with P' (contradiction). Thus, $B^*(\mathcal{Z}_j)$ of $\mathcal{Z}_j$ contains at least $\Omega(l^2)$ blue points.

Hence, we can assume that P'_j does not contain any red point. In this case, by putting z very close to $b_j b_{j+1}$, so that $z \in b_0 b_1 b_2$, we can make sure, that the corresponding wedge above the line $b_j b_{j+1}$ contains all the blue points in P'.

Thus, in every P_j, $0 \leq j \leq 2$, we have $\mathcal{Z}_j$ with b_j and b_{j+1} as $x(\mathcal{Z}_j)$ and $y(\mathcal{Z}_j)$, respectively, the set of blue points in P_j as $B(\mathcal{Z}_j)$, and the set of red points in P_j as $R(\mathcal{Z}_j)$. By using Proposition 9 on a nice partition of $\mathcal{Z}_j$ obtained by Lemma 11 we obtain a polygonal path using all the blue points in P_j which joins b_j and b_{j+1}, and which has all the red points in P_j on the "good" side. Finally, the required polygonization is obtained by concatenating the paths obtained by Lemma 11 (see Figure 5). $\square$

4 Concluding remarks

Theorem 1 in Section 3 proves the existence of a total blue polygonization excluding red points if we have enough inner blue points. We showed an upper bound on $K(l)$, the needed number of inner blue points, that is polynomial, but likely not tight. We conjecture that the upper bound is $2l-1$, which meets the lower bound in Theorem 2. If $l \leq 2$ then a non-trivial case-analysis shows that the conjecture holds. If finding the right values of $K(l)$ for all l turns out to be out of reach, it is natural to ask the following.

Question 1 *What is the right order of magnitude of $K(l)$?*

One could obtain a better upper bound on $K(l)$, e.g., by proving Lemma 11 with a weaker requirement on the number of blue points in $W(\mathcal{Z})$, which we suspect is possible.

Question 2 *Does Lemma 11 still hold, if we require only to have $\Omega(l)$ points in $W(\mathcal{Z})$, instead of $\Omega(l^2)$?*

Finally, the bounds we have on the minimal number of sides for the red-blue separating polygon do not meet.

Problem 1 *Improve the bounds n or/and $3\lceil n/2 \rceil$ in Theorem 4.*

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