New Lower Bounds for the Three-dimensional Orthogonal Bin Packing Problem*

Chia-Hong Hsu† Chung-Shou Liao§,†

Abstract

In this paper, we consider the three-dimensional orthogonal bin packing problem, which is a generalization of the well-known bin packing problem. We present new lower bounds for the problem and demonstrate that they improve the best previous results.

1 Introduction

The bin packing problem (abbreviated as 1D-BP) is one of the classic NP-hard combinatorial optimization problems. Given a set of n items with positive sizes v_1, v_2, $\ldots$, $v_n \leq B$, the objective is to find a packing in bins of equal capacity B to minimize the number of bins required. The problem finds obvious practical relevance in many industrial applications, such as the container loading problem and the cutting stock problem.

The bin packing problem is strongly NP-hard. Furthermore, it does not admit a $(\frac{3}{2}-\epsilon)$-factor approximation algorithm unless P=NP [10]. On the other hand, it has been shown that the simple *First Fit* approach can obtain a $\frac{17}{10}$-factor approximation algorithm, and the *First Fit Decreasing* algorithm can approximate within an asymptotic $\frac{11}{9}$-factor [11]. Subsequently, Fernandez de la Vega and Lueker [9] proposed an asymptotic polynomial time approximation scheme (PTAS), and Karmarkar and Karp [12] presented an improved asymptotic fully PTAS. For further results on approximation algorithms, readers may refer to Coffman, Garey, and Johnson's survey [6].

There are many variations of the bin packing problem, such as the strip packing, square packing, and rectangular box packing problems. In this paper, we consider the three-dimensional orthogonal bin packing problem (abbreviated as 3D-BP). Given an instance I of n 3D rectangular items I_1, I_2, $\ldots$, I_n, each item I_i is characterized by its width w_i, height h_i, depth d_i, and volume $v_i = w_i h_i d_i$. The goal is to determine a non-overlapping axis-parallel packing in identical 3D rectangular bins with width W, height H, depth D, and size $B = WHD$ that minimizes the number of bins required. We assume that the orientation of the given items is fixed; that is, the items cannot be rotated and they are packed with each side parallel to the corresponding bin side.

A considerable amount of research has been devoted to the design and analysis of lower bounds for the bin packing problem [4, 16, 22]. Martello and Toth [19, 20] and Labbé *et al.* [14] proposed lower bounds for 1D-BP, and then extended the concept to multi-dimensional models [17, 18]. Fekete and Schepers [7, 8] devised lower bounds based on *dual feasible functions* (please see the Appendix) and several related results were presented [3, 5]. Boschetti [1] combined Martello and Toth's work with the above dual feasible functions and proposed the best lower bound for 3D-BP; i.e., the lower bound *dominates*[1] all the previous results for 3D-BP.

In the following sections, we first review the previously proposed lower bounds and integrate the best of them for 1D-BP and 3D-BP to obtain a new lower bound for 3D-BP. Then, we propose another novel lower bound for 3D-BP and show that it dominates all the previous results.

2 Lower bounds for 1D-BP revisited

An obvious lower bound for 1D-BP, called the *continuous lower bound*, can be computed as follows:

$$L_0 = \left\lceil \frac{\Sigma_{i=1}^{n} v_i}{B} \right\rceil$$

It is known that the asymptotic worst-case performance ratio of the continuous lower bound L_0 is $\frac{1}{2}$ for 1D-BP [19]. The lower bound can be easily extended to 3D-BP by considering the volume v_i of each item I_i. Martello *et al.* [17] showed that, for 3D-BP, the worst-case performance ratio of L_0 is $\frac{1}{8}$.

Subsequently, the bound was improved by Martello and Toth [20]. Under the new bound denoted by L_1, the set of items is partitioned into two subsets such that one contains the items whose size is larger than $B/2$, and the other contains the remainder. Since each item in the former subset needs one bin, at least $\mid V(B/2, B] \mid$[2]

*Supported by the National Science Council of Taiwan under Grant NSC99-2218-E-007-010.

†Department of Industrial Engineering and Engineering Management, National Tsing Hua University, Hsinchu 300, Taiwan.
§Corresponding Author: csliao@ie.nthu.edu.tw

[1]For two lower bounds L_1 and L_2 of a minimization problem, L_2 is said to *dominate* L_1, denoted by $L_1 \leq L_2$, if for any instance I, $L_1(I) \leq L_2(I)$, where $L(I)$ is the value provided by a lower bound L for an instance I.

[2]For convenience, we define $V(a,b] = \{I_i \mid a < v_i \leq b\}$ and its cardinality as $\mid V(a,b] \mid$.

bins are required. In addition, only items of size v_i, $p \le v_i \le B - p$ are considered, where p is an integer with $1 \le p \le B/2$, because an item of size p cannot be placed in the same bin as an item whose size is greater than $B - p$. Hence, a valid lower bound L_1 can be computed if we allow the rest of the items (i.e., the items in $V[p, B/2]$) to be split. (The other rounding scheme of L_1, denoted by $L'_1(p)$, is described in the Appendix.)

Labbé *et al.* [14] further improved L_1, denoted as L_2, by partitioning the set of items into three subsets ($V(B/2, B]$, $V(B/3, B/2]$, and $V[p, B/3]$, where $1 \le p \le B/3$) and applying the *First Fit Decreasing* algorithm [6, 11, 13]. The procedure is implemented as follows. The items in $V(B/2, B]$ are assigned to separate bins as L_1. It may be possible to assign some of the items in $V(B/3, B/2]$ to the open bins, and at most one item in $V(B/3, B/2]$ can fit in any of the open bins. Thus, the open bins are sorted in non-decreasing order based on their residual space, and the items in $V(B/3, B/2]$ are assigned in non-decreasing order of their size. The procedure proves that the items in $V(B/2, B]$ and $V(B/3, B/2]$ can be matched optimally in a pairwise manner. Let K be the subset of items in $V(B/3, B/2]$ that cannot be matched through the above procedure. The items in K can be paired, so at least $\lceil K/2 \rceil$ bins are required. It follows that a valid lower bound L_2 can be obtained by allowing the items in $V[p, B/3]$ to be split as follows.

$$L_2 = \mid V(B/2, B] \mid + \lceil K/2 \rceil + \max_{1 \le p \le B/3} \{0, L_2(p)\}, \text{ where}$$

$$L_2(p) = \left\lceil \frac{\sum_{v_i \in V[p, B-p]} v_i}{B} - \mid V(B/2, B-p] \mid - \lceil K/2 \rceil \right\rceil$$

The lower bound L_2 can be obtained in $O(n)$ time provided that the sizes of the items are given sorted. Bourjolly and Rebetez [2] proved that $L_1 \le L_2$ (excluding the rounding scheme $L'_1(p)$), and that the asymptotic worst-case performance ratio of L_2 for 1D-BP is $\frac{3}{4}$. Note that the primal concept of Labbé *et al.* cannot be easily extended to a new lower bound L_{m-1} for 1D-BP by partitioning the set of items into m subsets[3], even by using a brute-force approach.

3 Lower bounds for 3D-BP revisited

For 3D-BP, Boschetti [1] proposed a lower bound, denoted by L_B, which actually consists of three types of lower bounds: $L_B(p, q, r)$, $L'_B(p, q, r)$, and $L''_B(p, q, r)$. We discuss them in detail below. Note that no dominance relations hold between the three bounds.

$$L_B = \max_{\substack{1 \le p \le W/2 \\ 1 \le q \le H/2 \\ 1 \le r \le D/2}} \{L_B(p, q, r), L'_B(p, q, r), L''_B(p, q, r)\}$$

Boschetti [1] proved that L_B is currently the best lower bound for 3D-BP by applying L_1 to $L_B(p, q, r)$ and $L'_B(p, q, r)$, denoted by $L_{B,1}$. In this section, we first review $L_{B,1}$ by applying L_1 to $L_B(p, q, r)$ and $L'_B(p, q, r)$. Then, based on the proofs in [2] and [3], which show, respectively, that $L_1 \le L_2$ and $L'_1(p) \le f_2^p$ (the *dual feasible function* f_2^p is discussed in the Appendix), we integrate L_2 in [14] and f_2^p in [3] with L_B to obtain a better lower bound for 3D-BP, denoted by $L_{B,2}$, and show that $L_{B,1} \le L_{B,2}$.

The lower bound $L_{B,2}(p, q, r)$. First, we consider the lower bound $L_B(p, q, r)$. Given an item $I_i = (w_i, h_i, d_i)$ for every i, we let $I^W(W - p, W] = \{I_i \mid W - p < w_i \le W\}$, $I^H(H - q, H] = \{I_i \mid H - q < h_i \le H\}$, $I^D(D - r, D] = \{I_i \mid D - r < d_i \le D\}$, and $I[p, q, r] = \{I_i \mid w_i \ge p,\ h_i \ge q,\ d_i \ge r\}$. The objective of $L_B(p, q, r)$ is to compute a valid lower bound for 1D-BP by using a simple rounding technique when considering the volume of each item in $I[p, q, r]$. For example, if $I_i \in I^W(W - p, W]$ is placed in a bin, then it will occupy a volume equal to $W h_i d_i$ since no items in $I[p, q, r]$ can be packed side by side parallel to the width. Hence, let $B = WHD$ and $L_B(p, q, r)$ be computed as a continuous lower bound by rounding the volume of each item v_i for every i to $v_i(p, q, r) = w_i(p) h_i(q) d_i(r)$ such that if $I_i \in I^W(W - p, W]$, i.e., $w_i > W - p$, then $w_i(p) = W$; otherwise, $w_i(p) = w_i$. If $I_i \in I^H(H - q, H]$, then $h_i(q) = H$; otherwise, $h_i(q) = h_i$. Similarly, if $I_i \in I^D(D - r, D]$, then $d_i(r) = D$; otherwise, $d_i(r) = d_i$. Note that it can be proved that this rounding technique is a *dual feasible function* [3, 8] (the so-called classic dual feasible function f_0^p; please see the Appendix).

$$L_B(p, q, r) = \left\lceil \frac{\Sigma_{i=1}^n v_i(p, q, r)}{B} \right\rceil$$

Since $L_B(p, q, r)$ can be computed as a continuous lower bound for 1D-BP by considering the volume of each item, L_1 can be applied to $L_B(p, q, r)$ to obtain a valid lower bound, denoted by $L_{B,1}(p, q, r)$. By contrast, we apply L_2 and the dual feasible function f_2^p to $L_B(p, q, r)$ separately. We select the maximum of the two refined lower bounds, denoted by $L_{B,2}(p, q, r)$, and show that it is a valid lower bound and that it is not smaller than $L_{B,1}(p, q, r)$.

Lemma 1 *$L_{B,2}(p, q, r)$ is a valid lower bound for 3D-BP, and it dominates $L_{B,1}(p, q, r)$.*

Proof. Based on the above rounding scheme, each item in $V(B/2, B]$ that is rounded, say w_i is rounded to W if

[3] Scholl *et al.* [21] showed that the lower bound L_2 can be extended by considering the items in $V(B/4, B/3]$, but the process is quite complicated and it does not have any obvious extension.

$w_i > W - p$, has two other dimensions larger than $H/2$ and $D/2$; otherwise, $v_i(p, q, r) \le B/2$. Hence, the items in $V(B/2, B]$ are assigned to separate bins.

Consider the items in $V(B/3, B/2]$. Assume we fit item I_i in $V(B/3, B/2]$ in an open bin, and item I_j in $V(B/2, B]$ is placed in the same bin. In addition, suppose the original dimensions of I_j are $w_j > W - p$, $h_j > H/2$, and $d_j > D/2$. Then, we only need to determine the height and depth of I_i since only the items in $I[p, q, r]$ are considered. We have $h_i > H/3$ and $d_i > D/3$ because $v_i(p, q, r) > B/3$. If $h_i < H/2$, it implies that $d_i > 2D/3$; similarly, if $d_i < D/2$, it implies that $h_i > 2H/3$. Thus, at most one item in $V(B/3, B/2]$ can fit in any of the open bins. The rounded items in $V(B/3, B/2]$ that can not be matched could not be matched originally either. Moreover, based on the above discussion, at most two items in $V(B/3, B/2]$ can be paired. Hence, it is valid to apply L_2 to $L_B(p, q, r)$.

Furthermore, $L_{B,2}(p, q, r)$ is a valid lower bound for 3D-BP because f_2^p is a dual feasible function, where $1 \le p \le B/2$ and it can be applied directly to $L_B(p, q, r)$. Because $L_1 \le L_2$ and $L_1'(p) \le f_2^p$, $L_{B,2}(p, q, r)$ dominates $L_{B,1}(p, q, r)$. □

The lower bound $L'_{B,2}(p, q, r)$. Regarding the lower bound $L'_B(p, q, r)$, as above, only the items in $I[p, q, r]$ are considered. Let $I(W - p, H - q, D - r) = I^W(W - p, W] \cap I^H(H - q, H] \cap I^D(D - r, D]$. Obviously, $\mid I(W - p, H - q, D - r) \mid$ is a valid lower bound and no items in $I[p, q, r]$ can be placed in the open bins. Next, the items in $I[p, q, r] \setminus I(W - p, H - q, D - r)$, denoted by $I'[p, q, r]$ are considered. The objective of $L'_B(p, q, r)$ is to consider items in terms of their width, height, and depth. Let the respective subsets be:

$$I(p, H - q, D - r) = I^H(H - q, H] \cap I^D(D - r, D] \cap I'[p, q, r];$$

$$I(W - p, q, D - r) = I^W(W - p, W] \cap I^D(D - r, D] \cap I'[p, q, r].$$

$$I(W - p, H - q, r) = I^W(W - p, W] \cap I^H(H - q, H] \cap I'[p, q, r];$$

Any two items from the different subsets above can not be matched in the same bin. That is, the items in $I(p, H - q, D - r)$, $I(W - p, q, D - r)$, and $I(W - p, H - q, r)$ can only be packed in separate bins. Thus, the items in $I(p, H - q, D - r)$, $I(W - p, q, D - r)$, and $I(W - p, H - q, r)$ can be considered separately. For each dimension, a continuous lower bound of 1D-BP can be computed similarly. It follows that a valid lower bound $L'_B(p, q, r)$ can be derived as follows:

$$L'_B(p, q, r) = \mid I(W - p, H - q, D - r) \mid +$$

$$\left\lceil \frac{\Sigma_{I_i \in I(p,H-q,D-r)} w_i}{W} \right\rceil + \left\lceil \frac{\Sigma_{I_i \in I(W-p,q,D-r)} h_i}{H} \right\rceil +$$

$$\left\lceil \frac{\Sigma_{I_i \in I(W-p,H-q,r)} d_i}{D} \right\rceil$$

Since a continuous lower bound of 1D-BP can be computed for each dimension, Boschetti [1] applied L_1 to the lower bound $L'_B(p, q, r)$, denoted by $L'_{B,1}(p, q, r)$, in terms of the width, height, and depth. Our lower bound, denoted as $L'_{B,2}(p, q, r)$, is obtained by applying L_2 and f_2^p to $L'_B(p, q, r)$ separately and selecting the maximum of the two refined lower bounds. We show that $L'_{B,2}(p, q, r)$ is still a valid lower bound.

Lemma 2 *$L'_{B,2}(p, q, r)$ is a valid lower bound for 3D-BP, and it dominates $L_{B,1}(p, q, r)$.*

Proof. Without loss of generality, we consider the depth of each item in $I(W - p, H - q, r) = I^W(W - p, W] \cap I^H(H - q, H] \cap I'[p, q, r]$. Because $r \le d_i < D - r$, the lower bound L_2 for 1D-BP can be used directly in terms of the depth of these items. This is similar to the width and height of the items in $I(p, H - q, D - r)$ and $I(W - p, q, D - r)$. Moreover, the dual feasible function f_2^p can be used directly for each dimension of the items. Hence, $L'_{B,2}(p, q, r)$ is a valid lower bound for 3D-BP. Because $L_1 \le L_2$ and $L_1'(p) \le f_2^p$, $L'_{B,2}(p, q, r)$ dominates $L'_{B,1}(p, q, r)$. □

The lower bound $L''_{B,2}(p, q, r)$. The lower bound $L''_B(p, q, r)$, which is conceptually similar to $L_B(p, q, r)$, can be obtained by using another rounding technique proposed in [18]. The objective is to pack items into a bin like small rectangular boxes whose dimensions are p, q, and r, where $1 \le p \le W/2$, $1 \le q \le H/2$, and $1 \le r \le D/2$. The maximum number of small rectangular boxes that can be placed in a bin is $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$. Besides, every item is represented by small rectangular boxes whose dimensions are p, q, and r. Thus, for every i, the volume of each item v_i, can be rounded to $v_i'(p, q, r) = w_i'(p)h_i'(q)d_i'(r)$ such that, if $I_i \in I^W(W/2, W]$, then $w_i'(p) = \lfloor W/p \rfloor - \lfloor (W - w_i)/p \rfloor$; otherwise, $w_i'(p) = \lfloor w_i/p \rfloor$. If $I_i \in I^H(H/2, H]$, then $h_i'(q) = \lfloor H/q \rfloor - \lfloor (H - h_i)/q \rfloor$; otherwise, $h_i'(q) = \lfloor h_i/q \rfloor$. Similarly, if $I_i \in I^D(D/2, D]$, then $d_i'(r) = \lfloor D/r \rfloor - \lfloor (D - d_i)/r \rfloor$; otherwise, $d_i'(r) = \lfloor d_i/r \rfloor$. For each dimension, it can be proved that the rounding technique is a dual feasible function [3, 8]. More precisely, it is similar to the dual feasible function f_2^p except that $w_i = W/2$, $h_i = H/2$, and $d_i = D/2$. $L''_B(p, q, r)$ can be computed as a continuous lower bound as follows:

$$L''_B(p, q, r) = \max_{\substack{1 \le p \le W/2 \\ 1 \le q \le H/2 \\ 1 \le r \le D/2}} \left\{ \left\lceil \frac{\Sigma_{i=1}^n v_i'(p, q, r)}{\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor} \right\rceil \right\}$$

We let the size of a bin B be equal to $\lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$ and apply L_2 to $L''_B(p, q, r)$, denoted by $L''_{B,2}(p, q, r)$, and show that it is also a valid lower bound.

Lemma 3 *$L''_{B,2}(p, q, r)$ is a valid lower bound for 3D-BP, and it dominates $L''_B(p, q, r)$.*

Proof. When L_2 is applied to $L''_B(p,q,r)$, the dimensions of the items in $V(B/2,B]$ are larger than $\frac{1}{2}\lfloor W/p\rfloor$, $\frac{1}{2}\lfloor H/q\rfloor$, and $\frac{1}{2}\lfloor D/r\rfloor$. The width w_i of each item I_i with $w'_i(p) > \frac{1}{2}\lfloor W/p\rfloor$ is larger than $W/2$ originally. Similarly, $h_i > H/2$ and $d_i > D/2$. Hence, the items in $V(B/2,B]$ are assigned to separate bins.

Consider each item I_i in $V(B/3,B/2]$. We have $w'_i(p) > \frac{1}{3}\lfloor W/p\rfloor$, $h'_i(q) > \frac{1}{3}\lfloor H/q\rfloor$, and $d'_i(r) > \frac{1}{3}\lfloor D/r\rfloor$, which implies that $w_i > W/3$, $h_i > H/3$, and $d_i > D/3$, because $v'_i(p,q,r) > B/3$. If item I_i can fit in an open bin, without loss of generality, there is one dimension of I_i, say $d'_i(r)$, that satisfies the condition $\frac{1}{2}\lfloor D/r\rfloor > d'_i(r) > \frac{1}{3}\lfloor D/r\rfloor$, which implies that $D/2 \geq d_i > D/3$. Furthermore, if $\frac{1}{2}\lfloor D/r\rfloor > d'_i(r)$, we have $w'_i(p) > \frac{2}{3}\lfloor W/p\rfloor$ and $h'_i(q) > \frac{2}{3}\lfloor H/q\rfloor$, which implies that $w_i > 2W/3$ and $h_i > 2H/3$ because $v'_i(p,q,r) > B/3$. Thus, at most one item in $V(B/3,B/2]$ can fit in any of the open bins; and at most two items in $V(B/3,B/2]$ can be paired.

On the other hand, we claim that if we can not fit I_i in some open bin, in which I_j in $V(B/2,B]$ is placed, then I_j was not matched with I_i originally. More precisely, if $d'_j(r)+d'_i(r) > \lfloor D/r\rfloor$, then $d_j+d_i > D$. We know that $d'_j(r) = \lfloor D/r\rfloor - \lfloor (D-d_j)/r\rfloor$. Suppose that $d_i \leq D/2$. Then, we have:

$$\left\lfloor\frac{D}{r}\right\rfloor - \left\lfloor\frac{D-d_j}{r}\right\rfloor + \left\lfloor\frac{d_i}{r}\right\rfloor > \left\lfloor\frac{D}{r}\right\rfloor$$

$$\Rightarrow \left\lfloor\frac{d_i}{r}\right\rfloor > \left\lfloor\frac{D-d_j}{r}\right\rfloor$$

$$\Rightarrow \frac{d_i}{r} \geq \left\lfloor\frac{d_i}{r}\right\rfloor \geq \left\lfloor\frac{D-d_j}{r}\right\rfloor + 1 > \left(\frac{D-d_j}{r} - 1\right) + 1$$

$$\Rightarrow d_j + d_i > D$$

We also know that $d_j + d_i > D$ if $d_i > D/2$; therefore, $L''_{B,2}(p,q,r)$ is a valid lower bound. Because $L_0 \leq L_2$, $L''_{B,2}(p,q,r)$ dominates $L''_B(p,q,r)$. □

Thus, we have the following new lower bound $L_{B,2}$ for 3D-BP:

$$L_{B,2} = \max_{\substack{1\leq p\leq W/3\\ 1\leq q\leq H/3\\ 1\leq r\leq D/3}} \{L_{B,2}(p,q,r), L'_{B,2}(p,q,r), L''_{B,2}(p,q,r)\}$$

The theorem follows immediately.

Theorem 4 $L_B \leq L_{B,1} \leq L_{B,2}$.

4 A new lower bound for 3D-BP

In this section, we extend the approach in [14] to 3D-BP and propose a novel lower bound, denoted by L^*_B. First of all, we define some notations. Let $I^W(W/2,W] = \{I_i \mid W/2 < w_i \leq W\}$, $I^H(H/2,H] = \{I_i \mid H/2 < h_i \leq H\}$, and $I^D(D/2,D] = \{I_i \mid D/2 < d_i \leq D\}$. Similarly, let $I^W(W/3,W/2] = \{I_i \mid W/3 < w_i \leq W/2\}$, $I^H(H/3,H/2] = \{I_i \mid H/3 < h_i \leq H/2\}$, and $I^D(D/3,D/2] = \{I_i \mid D/3 < d_i \leq D/2\}$.

$$I(W/2,H/2,D/2) = I^W(W/2,W] \cap I^H(H/2,H] \cap I^D(D/2,D];$$
$$I(W/3,H/2,D/2) = I^H(H/2,H] \cap I^D(D/2,D] \cap I^W(W/3,W/2];$$
$$I(W/2,H/3,D/2) = I^W(W/2,W] \cap I^D(D/2,D] \cap I^H(H/3,H/2];$$
$$I(W/2,H/2,D/3) = I^W(W/2,W] \cap I^H(H/2,H] \cap I^D(D/3,D/2];$$

We compute the new lower bound as follows. The items in $I(W/2,H/2,D/2) \cap V(B/3,B]$ are assigned to separate bins because each dimension of the items in $I(W/2,H/2,D/2)$ is larger than half the size of its corresponding bin side. It may be possible to assign some of the items in $I(W/3,H/2,D/2) \cap V(B/3,B]$, $I(W/2,H/3,D/2) \cap V(B/3,B]$, and $I(W/2,H/2,D/3) \cap V(B/3,B]$ to the open bins; however, at most one item can fit in any of the open bins because only the items in $V(B/3,B]$ are considered.

In addition, an item from the above three subsets can only fit in the open bins if one of its dimensions is smaller than half the size of the corresponding bin side; i.e., such an item can be only packed in the open bins in terms of its width, height, and depth. We then partition the $\mid I(W/2,H/2,D/2) \cap V(B/3,B] \mid$ open bins into two subsets so that one subset contains the open bins whose residual space is smaller than $B/2$, and the other contains the remaining bins. Note that the items in the first subset have at least two dimensions that are more than $2/3$ of the size of the corresponding bin sides. Thus, the open bins in that subset can be divided into three parts based on the smallest dimension of their included items. Moreover, the bins in each part are sorted in non-increasing order based on the corresponding dimension. Therefore, the items in $I(W/3,H/2,D/2) \cap V(B/3,B]$, $I(W/2,H/3,D/2) \cap V(B/3,B]$, and $I(W/2,H/2,D/3) \cap V(B/3,B]$ must be assigned in non-decreasing order separately in terms of their width, height, and depth. Similar to the proof of Labbé *et al.* [14], the procedure proves that the items are matched optimally in a pairwise manner.

Next, the second subset of open bins are sorted in non-decreasing order based on their residual space, and the items that cannot be matched are mixed and assigned in non-decreasing order according to their volume. Let $K^{HD} \subseteq I(W/3,H/2,D/2) \cap V(B/3,B]$ be the subset of items that cannot be matched through the above process. Similarly, let $K^{WD} \subseteq I(W/2,H/3,D/2) \cap V(B/3,B]$ and $K^{WH} \subseteq I(W/2,H/2,D/3) \cap V(B/3,B]$ be the subsets of items that cannot be matched either. Note that any two items from the different subsets above can not be matched in the same bin because, without loss of generality, one dimension of each item I_i, say w_i, no larger than $W/2$ implies that $h_i > 2H/3$ and $d_i > 2D/3$. Hence, the items in K^{HD}, K^{WD}, and K^{WH} can only be paired separately, and at least $\lceil K^{HD}/2\rceil + \lceil K^{WD}/2\rceil + \lceil K^{WH}/2\rceil$ bins are required.

Then, we consider the remaining items in $I(p, H-q, D-r)$, $I(W-p, q, D-r)$, and $I(W-p, H-q, r)$. Similarly, any two items from the different subsets can not be matched in the same bin. Thus, the items can be only packed in terms of each dimension. First, the items are assigned to the above open bins by allowing the items to be split. Let $I'(p, H-q, D-r)$, $I'(W-p, q, D-r)$, and $I'(W-p, H-q, r)$ be the subsets of items that cannot be packed in the $\mid I(W/2, H/2, D/2) \cap V(B/3, B] \mid + \lceil K^{HD}/2\rceil + \lceil K^{WD}/2\rceil + \lceil K^{WH}/2\rceil$ open bins respectively. We compute a continuous lower bound of 1D-BP for each dimension. Finally, a valid lower bound can be obtained by allowing the rest of the items to be split as follows:

$$L^*_B = \mid I(W/2, H/2, D/2) \cap V(B/3, B] \mid + \quad (1)$$

$$\left\lceil \frac{K^{HD}}{2} \right\rceil + \left\lceil \frac{K^{WD}}{2} \right\rceil + \left\lceil \frac{K^{WH}}{2} \right\rceil + \quad (2)$$

$$\left\lceil \frac{\sum_{I_i \in I'(p,H-q,D-r)} w_i}{W} \right\rceil + \quad (3)$$

$$\left\lceil \frac{\sum_{I_i \in I'(W-p,q,D-r)} h_i}{H} \right\rceil + \quad (4)$$

$$\left\lceil \frac{\sum_{I_i \in I'(W-p,H-q,r)} d_i}{D} \right\rceil + \quad (5)$$

$$\max_{\substack{1 \le p \le W/3 \\ 1 \le q \le H/3 \\ 1 \le r \le D/3}} \{0, L^*_B(p,q,r)\}, \text{ where } L^*_B(p,q,r) =$$

$$\left\lceil \frac{\sum_{I_i \in I'[p,q,r]} v_i}{B} - \alpha + \mid I(W-p, H-q, D-r) \mid \right\rceil$$

and $\alpha = (1) + (2) + (3) + (4) + (5)$.

We use the rounding scheme, i.e., the dual feasible function f_0^p for each dimension of every item I_i, to derive a rounded volume $v_i(p,q,r) = w_i(p)h_i(q)d_i(r)$. Next, we show that 1) L^*_B is a valid lower bound; and 2) after applying the rounding scheme f_0^p, L^*_B dominates $\max_{1 \le p \le W/3, 1 \le q \le H/3, 1 \le r \le D/3}\{L_{B,2}(p,q,r), L'_{B,2}(p,q,r)\}$.

Lemma 5 *L^*_B is a valid lower bound.*

Proof. The dimensions of each item in $I(W/2, H/2, D/2)$ are more than half the size of the corresponding bin sides even if the item is rounded. Hence, the items in $I(W/2, H/2, D/2)$ are assigned to separate bins.

Consider the items in $I(W/3, H/2, D/2) \cap V(B/3, B]$, $I(W/2, H/3, D/2) \cap V(B/3, B]$, and $I(W/2, H/2, D/3) \cap V(B/3, B]$. Without loss of generality, say we fit item $I_i \in I(W/3, H/2, D/2) \cap V(B/3, B]$ into an open bin, and item I_j in $I(W/2, H/2, D/2) \cap V(B/3, B]$ is placed in the same bin. I_i may fit with respect to the width because $h_i(q) > H/2$ and $d_i(r) > D/2$ imply that $h_i > H/2$ and $d_i > D/2$. Besides, $w_i = w_i(p)$ because $W/2 \ge w_i > W/3$. $W/2 \ge w_i$ also implies that $h_i(q) > 2H/3$ and $d_i(r) > 2D/3$ because $v_i(p,q,r) > B/3$. Thus, if h_i is rounded, then $h_i > H - q$; otherwise, $h_i > 2H/3$. Similarly, $d_i > \min\{D - r, 2D/3\}$. Because only the items in $I[p,q,r]$ are considered, at most one item in the above three subsets (every item I_k in the subsets has $w_k > W/3$, $h_k > H/3$, and $d_k > D/3$) can fit in any of the open bins.

On the other hand, since I_i may fit (in terms of the width) into the bin in which I_j is placed, we need to consider if w_j is rounded (because $w_i = w_i(p)$). We know that the rounded w_j that can not be matched was not matched originally either. In addition, based on the above discussion, for item $I_i \in K^{HD}$, $W/2 \ge w_i > W/3$ implies that $h_i > \min\{H - q, 2H/3\}$ and $d_i > \min\{D - r, 2D/3\}$. Thus, two items from any two of K^{HD}, K^{WD}, and K^{WH} cannot be matched in the same bin; and at most two items from each subset can be paired.

Finally, similar to the lower bound $L'_B(p,q,r)$, we consider the remaining items in $I(W-p, H-q, r)$, $I(p, H-q, D-r)$, and $I(W-p, q, D-r)$. The items are first assigned to the above open bins by allowing the items to be split. Then, we compute a continuous lower bound of 1D-BP for each dimension of the remainder. Thus, f_0^p can be applied to L^*_B, and L^*_B becomes a valid lower bound for 3D-BP by allowing the rest of the items to be split. □

Lemma 6 *For each $1 \le p \le W/3$, $1 \le q \le H/3$, $1 \le r \le D/3$, L^*_B dominates $L_{B,2}(p,q,r)$ and $L'_{B,2}(p,q,r)$.*

Proof. First we consider $L_{B,2}(p,q,r)$. Since f_0^p is applied to both $L_{B,2}(p,q,r)$ and our new lower bound L^*_B, we claim that the new partition scheme is better than Labbé *et al.*'s method. For the first part, we have $I(W/2, H/2, D/2) \cap V(B/3, B]$ open bins compared to $V(B/2, B]$ bins. Every item $I_k \in V(B/2, B]$ has $w_k(p) > W/2$, $h_k(q) > H/2$, and $d_k(r) > D/2$; thus, $I_k \in I(W/2, H/2, D/2) \cap V(B/3, B]$. We have $V(B/2, B] \subseteq I(W/2, H/2, D/2) \cap V(B/3, B]$.

For the second part, each item $I_k \in V(B/3, B]$ has $w_k(p) > W/3$, $h_k(q) > H/3$, and $d_k(r) > D/3$. Besides, if one of the item's dimensions, say the width $w_k(p) \le W/2$, it implies that $h_i(q) > 2H/3$ and $d_i(r) > 2D/3$. We have $V(B/3, B] \subseteq I(W/2, H/2, D/2) \cup I(W/3, H/2, D/2) \cup I(W/2, H/3, D/2) \cup I(W/2, H/2, D/3)$. Therefore, $\mid V(B/2, B] \mid + \lceil K/2\rceil \le \mid I(W/2, H/2, D/2) \cap V(B/3, B] \mid + \lceil K^{WH}/2\rceil + \lceil K^{HD}/2\rceil + \lceil K^{WD}/2\rceil$. It is obvious that the remainder of $L_{B,2}(p,q,r)$ is no larger than the remainder of L_B^*. Thus, L_B^* dominates $L_{B,2}(p,q,r)$.

Consider the lower bound $L'_{B,2}(p,q,r)$. For the first part, since f_0^p is applied to L_B^*, we have $I(W-p, H-q, D-r) \subseteq I(W/2, H/2, D/2) \cap V(B/3, B]$. Regarding the second part, without loss of generality, say $I(W-p, H-q, r)$ is considered in $L'_{B,2}(p,q,r)$. We explore the possibility of placing the items in $I(W/2, H/2, D/2) \cup I(W/2, H/2, D/3) \cup I(W-p, H-q, r)$ for the new lower bound L_B^*. Clearly, by considering each dimension, L_B^* dominates $L'_{B,2}(p,q,r)$. □

Finally, similar to $L''_{B,2}(p,q,r)$, we apply the dual feasible function f_2^p to each dimension of all the items instead. Then, we compute the summation of the rounded volume of each item, and a continuous lower bound can be obtained by letting the size of a bin $B = \lfloor W/p \rfloor \lfloor H/q \rfloor \lfloor D/r \rfloor$. It is also valid to apply L_2 to this continuous lower bound, denoted by $L_{DF}^*(p,q,r)$. Then, we have:

$$L_{B,DF}^* = \max\{L_B^*, L_{DF}^*(p,q,r)\}$$

Because $L''_{B,2}(p,q,r) \leq L_{DF}^*(p,q,r)$, the next theorem follows immediately.

Theorem 7 $L_{B,2} \leq L_{B,DF}^*$.

5 Concluding remarks

We have considered the 3D-BP problem and proposed two new lower bounds $L_{B,2}$ and $L_{B,DF}^*$. In addition, we have demonstrated that the lower bounds improve the best previous results, and that $L_{B,DF}^*$ dominates all the other lower bounds for 3D-BP proposed in the literature. In our future research, we will continue to improve the non-oriented model, which allows items to be rotated.

References

[1] M.A. Boschetti. New lower bounds for the three-dimensional finite bin packing problem. *Discrete Applied Mathematics*, 140:241–58, 2004.

[2] JM. Bourjolly, V. Rebetez. An analysis of lower bounds procedures for the bin packing problem. *Computers & Operations Research*, 32:395–405, 2005.

[3] J. Carlier, F. Clautiaux, A. Moukrim. New reduction procedures and lower bounds for the two-dimensional bin packing problem with fixed orientation. *Computers & Operations Research*, 34:2223–2250, 2007.

[4] F. Clautiaux, C. Alves, J.V. de Carvalho. A survey of dual-feasible and superadditive functions. *Annals of Operations Research*, 179:317–342, 2010.

[5] F. Clautiaux, A. Moukrim, J. Carlier. New data-dependent dual-feasible functions and lower bounds for a two dimensional bin-packing problem. *International Journal of Production Research*, 47(2):537–560, 2009.

[6] E.G. Coffman Jr., M.R. Garey, D.S. Johnson. Approximation algorithms for bin-packing: a survey. In D.S. Hochbaum (ed.) *Approximation algorithms for NP-hard problems*, 46–93, PWS Publishing, Boston MA, 1997.

[7] S.P. Fekete, J. Schepers. New classes of fast lower bounds for bin packing problems. *Mathematics Programming*, 91:11–31, 2001.

[8] S.P. Fekete, J. Schepers. A general framework for bounds for higher-dimensional orthogonal packing problems. *Mathematical Methods of Operations Research*, 60:311–329, 2004.

[9] W. Fernandez de la Vega, G.S. Lueker. Bin packing can be solved within $1+\epsilon$ in linear time. *Combinatorica*, 1:349–355. 1981.

[10] M.R. Garey, D.S. Johnson. Computers and intractability: a guide to the theory of NP-completeness. W.H. Freeman and Co., New York, 1979.

[11] D.S. Johnson. Near-optimal bin packing algorithms. Dissertation, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1973.

[12] N. Karmarkar, R.M. Karp. An efficient approximation scheme for the one-dimensional bin packing problem. In *Proc. 23rd IEEE Annu. Found. Comp. Sci. (FOCS 82)*, 312–320, 1982.

[13] B.H. Korte, J. Vygen. Combinatorial Optimization Theory and Algorithms (Chapter 18). Springer-Verlag, 2008.

[14] M. Labbé, G. Laporte, H. Mercure. Capacitated vehicle routing on trees. *Operations Research*, 39:616–622, 1991.

[15] G.S. Lueker. Bin packing with items uniformly distributed over intervals $[a,b]$. In *Proc. 24th IEEE Annu. Found. Comp. Sci. (FOCS 83)*, 289–297, 1983.

[16] A. Lodi, S. Martello, M. Monaci. Two-dimensional packing problems: A survey. *European Journal of Operational Research*, 141:241–252, 2002.

[17] S. Martello, D. Pisinger, D. Vigo. The three-dimensional bin packing problem. *Operations Research*, 48(2):256–267, 2000.

[18] S. Martello, D. Vigo. Exact solution of the two-dimensional finite bin packing problem. *Management Science*, 44:388–399, 1998.

[19] S. Martello, P. Toth. Knapsack problems: algorithms and computer implementations. John Wiley & Sons, Chichester, U.K., 1990.

[20] S. Martello, P. Toth. Lower bounds and reduction procedures for the bin packing problem. *Discrete Applied Mathematics*, 28:59–70, 1990.

[21] A. Scholl, R. Klein, C. Jürgens. BISON: A fast hybrid procedure for exactly solving the one-dimensional bin packing problem. *Computers & Operations Research*, 24:627–645, 1997.

[22] S.S. Seiden, R. van Stee New bounds for multidimensional packing. *Algorithmica*, 36:261–293, 2003.