On the VC-Dimension of Visibility in Monotone Polygons

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Abstract

We show that the VC-dimension of visibility on the boundary of a monotone polygon is exactly 6. Our lower bound construction matches the best known lower bound for simple polygons.

1 Introduction

The *art gallery problem* is perhaps one of the best known problems in computational geometry. An instance of the art gallery problem takes as input a polygon P. The polygon P is defined by a set of points $V = \{v_1, v_2, \ldots, v_n\}$. There are edges connecting (v_i, v_{i+1}) where $i = 1, 2, \ldots, n-1$. There is an edge connecting (v_n, v_1). If these edges do not intersect other than at the points in V, then P is called a *simple polygon*. The edges of a simple polygon give us two disjoint regions: inside the polygon and outside the polygon. For any two points $p, q \in P$, we say that *p sees q* if the line segment $\overline{pq}$ does not go outside of P. The art gallery problem seeks to find a set of points $G \subseteq P$ such that every point $p \in P$ is seen by a point in G. We call this set G a guarding set. The optimization problem is thus defined as finding the smallest such G. Art gallery problems are motivated by applications such as line-of-sight transmission networks in terrains, signal communications and broadcasting, cellular telephony systems and other telecommunication technologies as well as placement of motion detectors and security cameras.

1.1 Previous Work on the Art Gallery Problem

The question of whether guarding simple polygons is NP-hard was confirmed by Aggarwal [1] and Lee and Lin [16] independently roughly thirty years ago. They showed that the problem is NP-hard for both vertex guards (where one can only chose points in V to be guards) and interior guards (guards can be anywhere inside P). Along with being NP-complete, Brodén *et al.* and Eidenbenz [2, 6] independently prove that interior guarding simple polygons is APX-hard. This means that there exists a constant $\epsilon > 0$ such that no polynomial-time algorithm can guarantee an approximation ratio of $(1 + \epsilon)$ unless P=NP.

Ghosh provides a $O(\log n)$-approximation for the problem of vertex guarding an n-vertex simple polygon in [7]. This result can be improved for simple polygons using randomization, giving an algorithm with expected running time $O(nOPT_v^2 \log^4 n)$ that produces a vertex guard cover with approximation factor $O(\log OPT_v)$ with high probability, where OPT_v is the smallest vertex guard cover for the polygon [5]. Whether a polynomial-time constant factor approximation algorithm can be obtained for vertex guarding a simple polygon is a long-standing and well-known open problem. Deshpande *et al.* [4] present a pseudopolynomial randomized algorithm for finding a point guard cover with approximation factor $O(\log OPT)$. King and Kirkpatrick provide an $O(\log \log OPT)$-approximation algorithm for the problem of guarding a simple polygon with guards on the perimeter in [12]. The point guarding problem seems to be much more difficult and precious little is known about it [4].

Due to the inherent difficulty in fully understanding the art gallery problem for simple polygons, there has been some work done guarding polygons with some additional structure. Krohn and Nilsson [15] give a polynomial-time constant factor approximation algorithm for the special case of the problem when the polygon is x-monotone. They also proved point guarding and vertex guarding a monotone polygon is NP-hard [14]. A polygon P is *x-monotone* (or simply *monotone*) if any vertical line intersects the boundary of P in at most two points. Let a and b denote the leftmost and rightmost point of P respectively. Consider the "top half" of the boundary of P by walking along the boundary "clockwise" from a to b. We call this the *ceiling* of P. Similarly we obtain the *floor* of P by walking "clockwise" along the boundary from b to a. Notice that both the ceiling and the floor are x-monotone polygonal chains, that is a vertical line intersects it in at most one point. Guarding a monotone polygon has some similarities with the well-studied *terrain guarding problem*, where we are interested in guarding an x-monotone polygonal chain. Gibson *et al.* [8] present a polynomial-time approximation scheme for the terrain guarding problem improving upon several recent constant factor approximations, and King and Krohn proved that the problem is NP-hard [13].

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1.2 VC-Dimension

An interesting measure of the complexity of a set system is the notion of *VC-dimension*. To define this, we say that a finite set of points G in P is *shattered* if for every subset of $G' \subseteq G$ there exists some point $v \in P$ such that v sees every point in G' and does not see any point in $G \setminus G'$. In this context, we call v a *viewpoint*. The VC-dimension is the largest d such that there exists some polygon P and point set G of size d that can be shattered.

Brönnimann and Goodrich give a polynomial-time $O(\log OPT)$-approximation algorithm for any set system with constant VC-dimension [3]. In 1998, Valtr showed that the VC-dimension of the visiblity in a simple polygon is between 6 and 23 [17]. The lower bound of 6 still to this day is the best known lower bound, and the upper bound was not improved until very recently by Gilbers and Klein who give an upper bound of 14 [10]. They also suggest that the actual VC-dimension is likely to be closer to the lower bound of 6 rather than the upper bound of 14. The upper bound of 14 for simple polygons also applies to monotone polygons, but the lower bound construction of 6 given by Valtr is not a monotone polygon and therefore does not apply to monotone polygons. King [11] proved that the VC-dimension of visibility on x-monotone terrains is exactly 4, and due to the relationship of a terrain and the floor of a monotone polygon, this lower bound of 4 can easily be extended to obtain a lower bound of 4 for monotone polygons. Up until very recently, the best known bounds on the VC-dimension of visiblity in the boundary of monotone polygons are 4 and 14. In a currently-unpublished result [9], Gilbers proved an upper bound of 7 on the VC-dimension of visibility in the boundary of a simple polygon.

1.3 Our Contribution

In this paper, we prove the following theorem.

Theorem 1 *The VC-dimension of the visibility on the boundary of a monotone polygon is 6.*

We improve the lower bound from 4 to 6 by showing that there is a set of 6 points on the boundary of some monotone polygon that can be shattered by a set of 2^6 viewpoints on the boundary of the polygon, and we improve the upper bound from 7 to 6 by showing that any set of 7 or more points on the boundary of a monotone polygon cannot be shattered by 2^7 viewpoints on the boundary of the polygon. Note that our lower bound result matches the best known lower bound result for simple polygons. Valtr's lower bound construction does not place all of the viewpoints on the boundary of the polygon, and therefore our result improves the best known lower bound for the VC-dimension of visibility in the boundary of simple polygons as well.

1.4 Organization of the Paper

In Section 2, we provide some definitions and key observations regarding the visibility of points on the boundary of a monotone polygon. In Section 3, we prove that the VC-dimension of the visibility in the boundary of a monotone polygon is 6.

2 Preliminaries

In this section, we provide some preliminaries needed before giving the details of the proof of Theorem 1. We begin with some definitions, and then give some key lemmas utilized in our proof.

Definitions Recall the definition of ceiling and floor of P given in the introduction. We say two points on the boundary are *on the same side* of P if they are both on the ceiling or if they are both on the floor. If the x-coordinate of a point s in P is less than the x-coordinate of a point t in P then we say *s is to the left of t* or *t is to the right of s*, and we denote this $s < t$ or $t > s$.

We call the line segment connecting two points that see each other a *good line segment*. Consider two points a and b in a monotone polygon. We say that the line segment $\overline{ab}$ is *covered from above* if for all points $p \in \overline{ab}$, the ray shot directly up from p intersects a good line segment. Similarly, we say that $\overline{ab}$ is *covered from below* if for all points $p \in \overline{ab}$, the ray shot directly down from p intersects a good line segment. If a line segment $\overline{ab}$ is covered by some good line segments from above and below, then we say this line segment is *sandwiched*. Note that if a line segment $\overline{ab}$ is sandwiched then a and b have to see each other. If they do not see each other, either there is a point on the floor above $\overline{ab}$ or a point on the ceiling below $\overline{ab}$. In either case, this point would block the endpoints of a good line segment from seeing each other, a contradiction. Therefore, if two points should not see each other, then the line segment connecting them cannot be sandwiched. See Figure 1 for an illustration.

Key Lemmas We now give some key observations that enabled us to construct our lower bound example and give the upper bound proof. For ease of description, the lemmas are not stated in their fullest generality, but it is not hard to see that the lemmas also apply in symmetric scenarios. See Figure 4 for an illustration of the lemmas.

A key property of terrains is characterized by the *order claim* which states the following: if we have four points a, b, c, and d on the terrain satisfying (1) $a < b < c < d$, (2) a sees c, and (3) b sees d, then

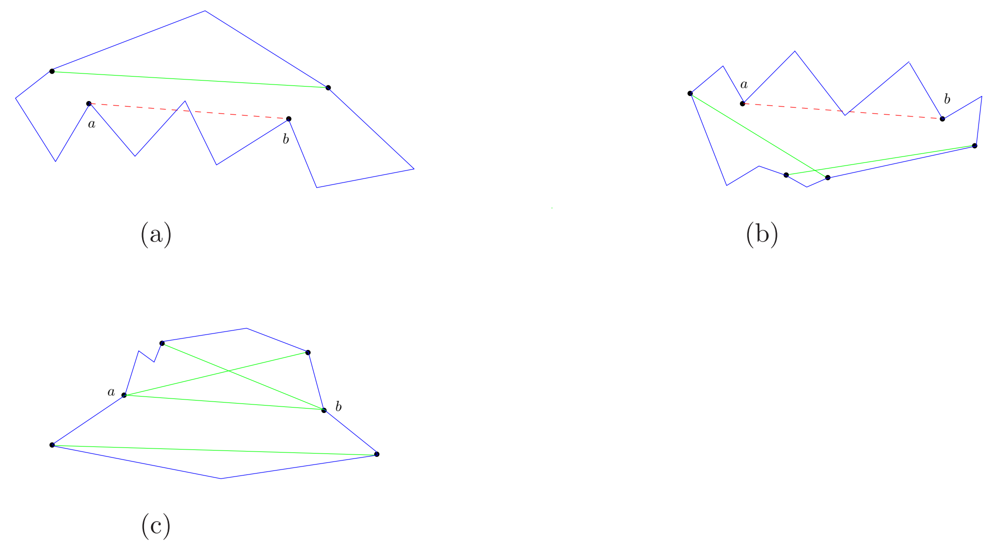


Figure 1: Part (a) $\overline{ab}$ is covered from above and blocked by the floor. Part (b) $\overline{ab}$ is covered from below and blocked by the ceiling. Part (c) $\overline{ab}$ is *sandwiched*

a sees d. Since the ceiling and floor individually are just terrains, it is natural to wonder if the order claim also applies to monotone polygons. It is not hard to see that it does not, as the ceiling can block a from seeing d without blocking $\overline{ac}$ or $\overline{bd}$ (see Figure 2); however, the following lemma shows that a similar concept does indeed apply if an additional condition holds.

Lemma 2 *Suppose a, b, c, and d are four points on the boundary of a monotone polygon P such that $a < b < c < d$. Suppose that a, b, and c are on the ceiling of the polygon (d can be either on the ceiling or floor), a sees c, and b sees d. If the line segment $\overline{ad}$ is covered from below, then a sees d.*

Proof. It is easy to see that $\overline{ac} \cup \overline{bd}$ covers $\overline{ad}$ from above, and by assumption $\overline{ad}$ is covered from below, so $\overline{ad}$ is sandwiched. See Figure 4 (a). $\square$

The following lemma also plays an important role.

Lemma 3 *Suppose a, b, and c are three points on the boundary of a monotone polygon P such that $a < b < c$. If c sees a and does not see b and exactly one of $\{b, c\}$ is on the ceiling then b cannot see any point to the right of c.*

Proof. Without loss of generality, assume that b is on the floor and c is on the ceiling. Since we have $a < b$, then $\overline{ac}$ covers $\overline{bc}$ from above. Now consider a point p to the right of c. If c is "below" the line segment $\overline{bp}$, then c blocks b from seeing p. If c is above $\overline{bp}$, then $\overline{bp}$ covers $\overline{bc}$ from below, and therefore $\overline{bc}$ is sandwiched, contradicting the assumption that b does not see c. See Figure 3 for an illustration. $\square$

An extension of Lemma 3 yields the following lemma.

Lemma 4 *Suppose a, b, c, d, and e are five points on the boundary of a monotone polygon P such that $a < b < c < d < e$. Suppose (1) b and d are on the same side of the polygon, (2) c is on the opposite side of the polygon, and (3) c sees a and e and does not see b and d. Then there is no point in P that sees both b and d.*

Proof. By Lemma 3, b does not see any point to the right of c and d does not see any point to the left of c, then there is no place to put a point that can see b and d. See Figure 4 (c). $\square$

Combining the ideas from Lemmas 2 and 3, we obtain the following lemma which plays a large role in our upper bound proof.

Lemma 5 *Let a, b, and c be three points on the boundary of a monotone polygon P such that $a < b < c$ and b and c are on the same side of P. Let p_1 and p_2 be points on the boundary of P such that p_1 sees a and c and does not see b, and p_2 sees b and does not see c. If $c < p_1$ and $c < p_2$, then it must be that $p_1 < p_2$, and p_1 must be on the same side of P as b and c.*

Proof. Without loss of generality, we assume that b and c are both on the ceiling. It is easy to see that if p_1 is on the ceiling to the left of p_2 then we can block b from seeing p_1 and block c from seeing p_2 without blocking a good line segment. See Figure 4 (d). We will now show that any other placement of p_1 and p_2 to the right of c will violate one of our previous lemmas. If we place p_2 on the ceiling to the left of p_1, then we have that b sees p_1 by Lemma 2, a contradiction. If we place p_2 on the floor to the left of p_1, then c cannot see to the right

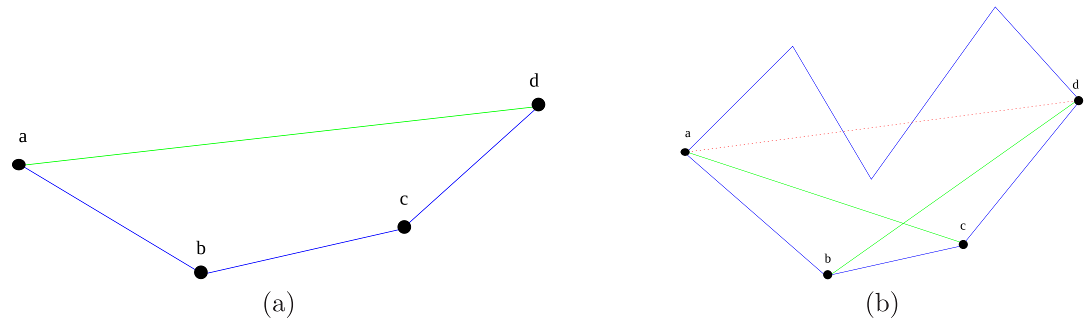


Figure 2: Part (a) An illustration of the order claim for terrains. Part (b) An illustration of monotone polygons.

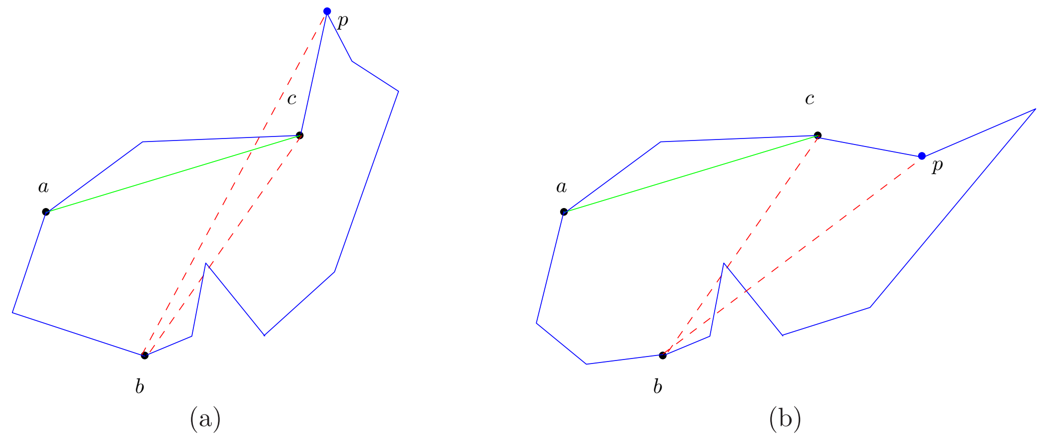


Figure 3: Part (a) c is "below" the line segment $\overline{bp}$. Part (b) c is "above" the line segment $\overline{bp}$.

of p_2 by Lemma 3, contradicting the assumption that c sees p_1. If we place p_1 on the floor to the left of p_2, then b cannot see any point to the right of p_1 by Lemma 3, contradicting the assumption that b sees p_2. □

3 VC-Dimension

In this section, we prove Theorem 1. Recall that best known lower bound on the VC-dimension of a simple polygon is also 6 [17], but the polygon in this construction is not monotone and many of the viewpoints lie in the interior of the polygon. We then show that for any 7 points on the boundary of a monotone polygon, it is not possible to shatter them by a set of 2^7 points on the boundary of the polygon.

3.1 Lower Bound

In this section we give our construction that shows that there is a set G of six points on the boundary of a monotone polygon that can be shattered. We feel that the result is somewhat surprising given that the best known lower bound on the VC-dimension of a simple polygon is also six. See Figure 5 for the construction. In the figure, points in G are black, and the viewpoints are red. The label on the viewpoints denotes which subset of G the viewpoint sees. We do not show the viewpoints that see at most one point of G. These can be added in steep "canyons" below each point of G for the viewpoints that see only one point, and the point that sees zero of the points can be handled similarly.

3.2 Upper Bound

We will now prove that any 7 points on the boundary of a monotone polygon cannot be shattered by a set of points on the boundary of the polygon. Our proof will use the lemmas from Section 2. Let G be any set of 7 points on the boundary of a monotone polygon P. For any subset $S \subseteq G$, we let v_S denote the viewpoint that sees each point in S and does not see any point in $G \setminus S$. Let g_ℓ be the point in G that is farthest to the left, and let g_r be the point in G that is farthest to the right (breaking ties arbitrarily in both cases). Let G' denote the other 5 points in G. Our proof considers three main cases based on the position of these points: (1) all five points are on the ceiling, (2) there are four points on the ceiling and one on the floor, and (3) there are three points on the ceiling and two on the floor. Note that any other situation (i.e. when there are more points from G' on the floor than on the ceiling) is symmetric to one of these three cases. In each case, we will make

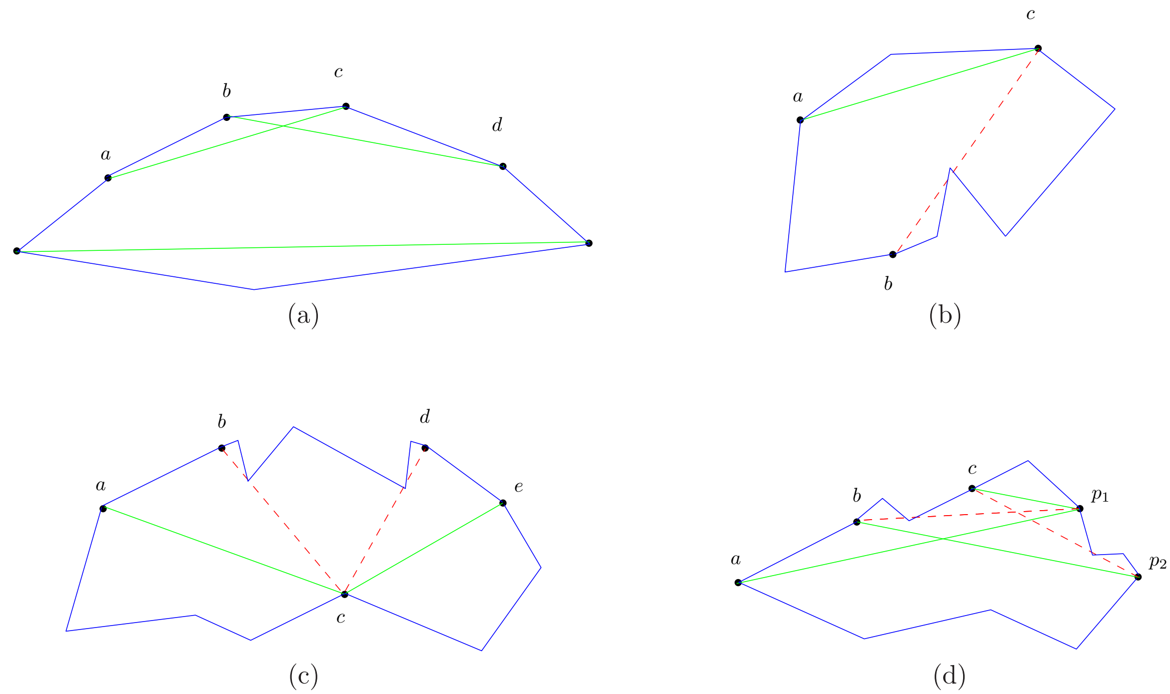


Figure 4: (a) An illustration of Lemma 2, (b) an illustration of Lemma 3, (c) an illustration of Lemma 4, and (d) an illustration of Lemma 5.

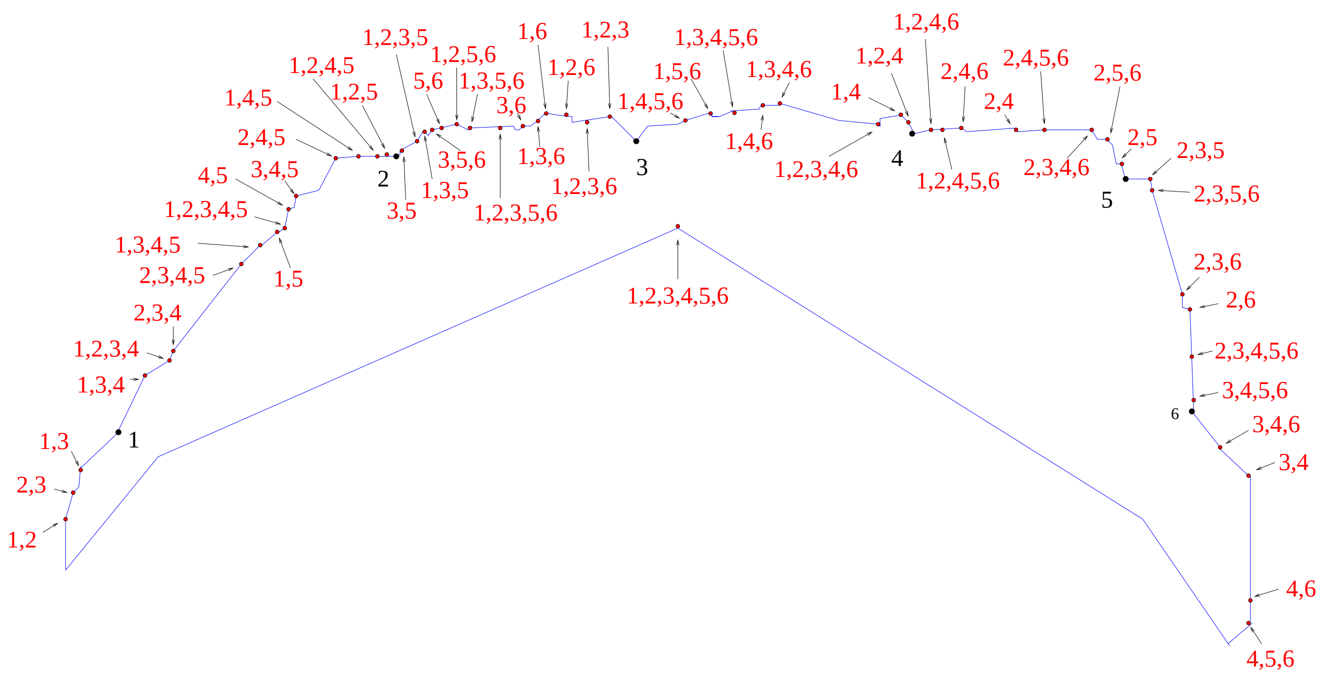


Figure 5: Lower Bound Construction

no assumption as to whether g_ℓ and g_r are on the ceiling or the floor.

3.2.1 Case 1.

In this case, all five of the points in G' are on the ceiling, and we denote the ith point in G' from left to right c_i. Consider the viewpoint $v_1 = vp_{\{g_\ell,c_2,c_4,g_r\}}$, and without loss of generality assume that it is to the left of c_3. Now consider the viewpoints $v_2 = vp_{\{c_1,c_3,c_5\}}$ and $v_3 = vp_{\{g_\ell,c_2,c_3,c_5,g_r\}}$. We will show that these three viewpoints cannot be placed together. We will first argue that v_2 and v_3 individually cannot be placed to the left of c_4, and then we will show that they cannot be simultaneously placed to the right of c_4.

We will argue that v_2 cannot go to the left of c_4, and it will be easy to see that the argument also holds for v_3. Placing v_2 between c_4 and v_1 violates Lemma 5 with $a = g_r, b = c_5, c = c_4, p_1 = v_1$, and $p_2 = v_2$ (see Figure 6 (a)). Placing v_2 to the left of v_1 violates Lemma 5 with $a = c_5, b = c_4, c = c_3, p_1 = v_2$, and $p_2 = v_1$ (see Figure 6 (b)). Therefore v_2 (and v_3) cannot go to the left of c_4. We will now show that v_2 and v_3 cannot simultaneously be placed to the right of c_4. Applying Lemma 5 with $a = g_\ell, b = c_1, c = c_2, p_1 = v_3$, and $p_2 = v_2$, we have v_3 must be on the ceiling to the left of v_2. But then we have that c_2 sees v_2 by Lemma 2 where $a = c_2, b = c_3, c = v_3$, and $d = v_2$(see Figure 6 (c)).

3.2.2 Case 2.

In this case, four of the points in G' are on the ceiling and there is one point in G' on the floor. We denote the ceiling points c_1, c_2, c_3, and c_4 from left to right and the floor point f_1. Consider the viewpoint $v_1 = vp_{\{g_\ell,c_1,c_3,g_r\}}$. We will consider two subcases based on the position of v_1.

v_1 **is to the left of** c_2. Consider the viewpoint $v_2 = vp_{\{g_\ell,c_2,c_4,g_r\}}$. We will show that v_2 cannot go to the left of c_3. If we place v_2 between c_3 and v_1 then we violate Lemma 5 with $a = g_r, b = c_4, c = c_3, p_1 = v_1$, and $p_2 = v_2$ (see Figure 7 (a)). If we place v_2 to the left of v_1 then we violate Lemma 5 with $a = g_r, b = c_3, c = c_2, p_1 = v_2$, and $p_2 = v_1$ (see Figure 7 (b)). Therefore v_2 cannot go to the left of c_3.

Now consider the viewpoint $v_3 = vp_{\{g_\ell,c_1,c_4,f_1,g_r\}}$. We will show that v_3 cannot go to the left of c_3. Applying Lemma 5 with $a = g_r, b = c_4, c = c_3, p_1 = v_1$, and $p_2 = v_3$, we can see that v_3 must be to the left of v_1 and v_1 must be on the ceiling (see Figure 7 (c)). Now we can see that f_1 cannot be to the right of v_1, because if it were then it couldn't see v_3 by Lemma 3. So suppose that f_1 is to the left of v_1. By Lemma 3, we have that f_1 cannot see any point to the right of v_1. But if the viewpoint $vp_{\{c_2,f_1,g_r\}}$ is to the left of v_1, then we have that c_3 will see it by Lemma 2 (see Figure 7 (d)).

So we now have that both v_2 and v_3 must go to the right of c_3. By Lemma 5 with $a = g_\ell, b = c_1, c = c_2, p_1 = v_2$, and $p_2 = v_3$ that v_2 is to the left of v_3 and is on the ceiling (see Figure 7 (e)). Now we can see that f_1 cannot be to the left of v_2, because if it were then it couldn't see v_3 by Lemma 3. So suppose that f_1 is to the right of v_2. By Lemma 3, we have that f_1 cannot see any point to the left of v_2. But if the viewpoint $vp_{\{g_\ell,c_3,f_1\}}$ is to the right of v_2, then we have that c_2 will see it by Lemma 2 (see Figure 7 (f)).

v_1 **is to the right of** c_2. Now consider the viewpoints $v_4 = vp_{\{g_\ell,c_2,c_3,g_r\}}$ and $v_5 = vp_{\{g_\ell,c_2,c_4,f_1,g_r\}}$. We will show that these points cannot go to the right of c_3. We will argue from the perspective of v_4 but the same arguments hold for v_5 as well. Suppose v_4 is to the right of c_3. Then by Lemma 5 with $a = g_\ell, b = c_1, c = c_2, p_1 = v_4$, and $p_2 = v_1$, it must be that $v_4 < v_1$ and that v_4 is on the ceiling. But this implies that c_2 sees v_1 by Lemma 2. Therefore it cannot be the case that v_4 (and v_5) is to the right of c_3 (see Figure 7 (g)).

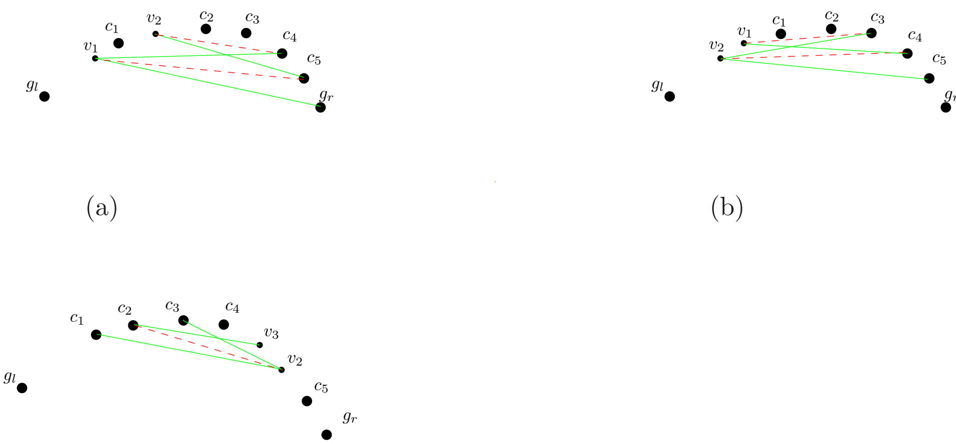
Now we will assume that v_4 and v_5 are to the left of c_3. By Lemma 5 with $a = g_r, b = c_4, c = c_3, p_1 = v_4$, and $p_2 = v_5$, we have that $v_5 < v_4$ and that v_4 must be on the ceiling. We can now see that f_1 cannot be to the right of v_4, because if it were then it could not see any point to the left of v_4 by Lemma 3 and therefore could not see v_5. So we will assume that f_1 is to the left of v_4, and therefore cannot see any point to the right of v_4 by Lemma 3 (see Figure 7 (h)).

So now consider the viewpoint $v_6 = vp_{\{g_\ell,c_1,c_3,f_1,g_r\}}$. Since f_1 sees both v_5 and v_6, they both must be to the left of v_4 and thus necessarily to the left of c_3. By Lemma 5 with $a = g_r, b = c_4, c = c_3, p_1 = v_6$, and $p_2 = v_5$, it must be that $v_5 < v_6$ and that v_6 is on the ceiling. If v_6 is to the right of c_2, then we have that c_1 sees v_4 by Lemma 2 (see Figure 7 (i)). If v_6 is to the left of c_2, then we have that c_3 sees v_5 by Lemma 2 (see Figure 7 (j)). This concludes the proof that v_1 cannot go to the right of c_2, and also completes the proof of Case 2.

3.2.3 Case 3.

In this case, we have three points of G' on the ceiling and two on the floor. Let c_1, c_2, and c_3 denote the three ceiling points from left to right, and similarly let f_1 and f_2 denote the two floor points. We make no assumption whether a ceiling point c_i is to the left or right of a floor point f_j.

Consider the set V of six viewpoints that see g_ℓ, g_r, exactly one ceiling point, and exactly one floor point (i.e. $vp_{\{g_\ell,c_2,f_2,g_r\}}$, $vp_{\{g_\ell,c_3,f_1,g_r\}}$, etc.). For simplicity, we denote the viewpoint that sees c_i and f_j as $v_{i,j}$. We



(a) (b) (c)

Figure 6: An illustration of Case 1.

will show that it is not possible to place all viewpoints in V along with the viewpoint $p = vp_{\{g_\ell, c_1, c_3, f_1, f_2, g_r\}}$. To do this, we will make use of the following corollary that easily follows from Lemma 3.

Corollary 6 *If there is a viewpoint v on the ceiling that sees g_ℓ and g_r and does not see some floor point f_i, then there cannot be any pair of viewpoints p_1 and p_2 that both see f_i such that $p_1 < v < p_2$.*

We begin by showing that it is not possible to place both $v_{2,1}$ and $v_{2,2}$ on the ceiling. Suppose that they are both on the ceiling, and assume without loss of generality that $v_{2,1} < v_{2,2}$. Now consider placing the point p. Note that by Corollary 6, it must be that p is between $v_{2,1}$ and $v_{2,2}$. Suppose we place p to the left of c_2, which implies that $v_{2,1}$ is also to the left of c_2. Applying Lemma 5 with $a = g_r, b = c_3, c = c_2, p_1 = v_{2,1}$ and $p_2 = p$, we see that p must be to the left of $v_{2,1}$, a contradiction (see Figure 8 (a)). A symmetric argument holds when p is placed between $v_{2,1}$ and $v_{2,2}$ to the right of c_2, and we conclude that at least one of $v_{2,1}$ and $v_{2,2}$ is on the floor.

So now we will assume that at least one of $v_{2,1}$ and $v_{2,2}$ is on the floor. For the remainder of the proof, we will refer to the floor points as f_i and f_j as the arguments will not depend on the left-to-right orientation of the floor points. So now consider placing the viewpoint $v_{2,i}$ on the floor. It cannot be placed between c_1 and c_3 by Lemma 4, and we assume it is to the left of c_1 without loss of generality (see Figure 8 (b)).

We will now show that at least one of c_1 and c_3 must have both of their corresponding points in V on the ceiling by showing that if a c_1 point in V is on the floor then a c_3 point in V cannot also be on the floor. Without loss of generality, suppose $v_{1,i}$ is on the floor. By Lemma 3, we have that $v_{1,i}$ cannot go to the left of $v_{2,i}$ as c_1 cannot see any points to the left of $v_{2,i}$ (see Figure 8 (c)), and it cannot go between $v_{2,i}$ and c_2 because c_2 would not be able to see $v_{2,i}$ by Lemma 3 (see Figure 8 (d)). Therefore if $v_{1,i}$ is on the floor, it must go to the right of c_2. It also cannot go between c_2 and c_3 by Lemma 4 (see Figure 8 (e)), and therefore must be to the right of c_3. Now without loss of generality consider placing $v_{3,j}$ on the floor. For the same reasoning as $v_{1,i}$, it cannot be placed to the left of c_2. It also cannot be placed between c_2 and $v_{1,i}$ or c_1 would not see $v_{1,i}$ by Lemma 3. Finally we cannot place $v_{3,j}$ to the right of $v_{1,i}$ as c_3 would not see it by Lemma 3 (see Figure 8 (f)).

We now have that at least one of c_1 and c_3 have both of their corresponding points in V on the ceiling. Note that in either case, there must be a point on the ceiling that sees g_ℓ and g_r and does not see f_i as well as a point on the ceiling that sees g_ℓ and g_r and does not see f_j. We conclude the proof of Case 3 by showing that in either scenario we cannot place all of the points in $V \cup \{p\}$.

$v_{1,i}$ and $v_{1,j}$ are both on the ceiling. First note that both $v_{1,i}$ and $v_{1,j}$ must be to the right of $v_{2,i}$ for c_1 to be able to see them, and therefore by Corollary 6 it must be that $v_{1,i} < v_{1,j}$. Now consider the placement of p. Corollary 6 implies p must be between $v_{1,i}$ and $v_{1,j}$ because p sees both f_i and f_j. Further note that p cannot be to the left of c_2 (see Figure 8 (g)), because Lemma 5 with $a = g_r, b = c_3, c = c_2, p_1 = v_{2,i}$, and $p_2 = p$ implies that $v_{2,i}$ would need to be on the ceiling, but $v_{2,i}$ is on the floor by assumption. This implies that p and $v_{1,j}$ both need to be to the right of c_2.

Now consider $v_{2,j}$. We will first show that it cannot go to the left of c_2. By Corollary 6, it cannot go to the left of $v_{1,i}$. It cannot go on the ceiling between $v_{1,i}$

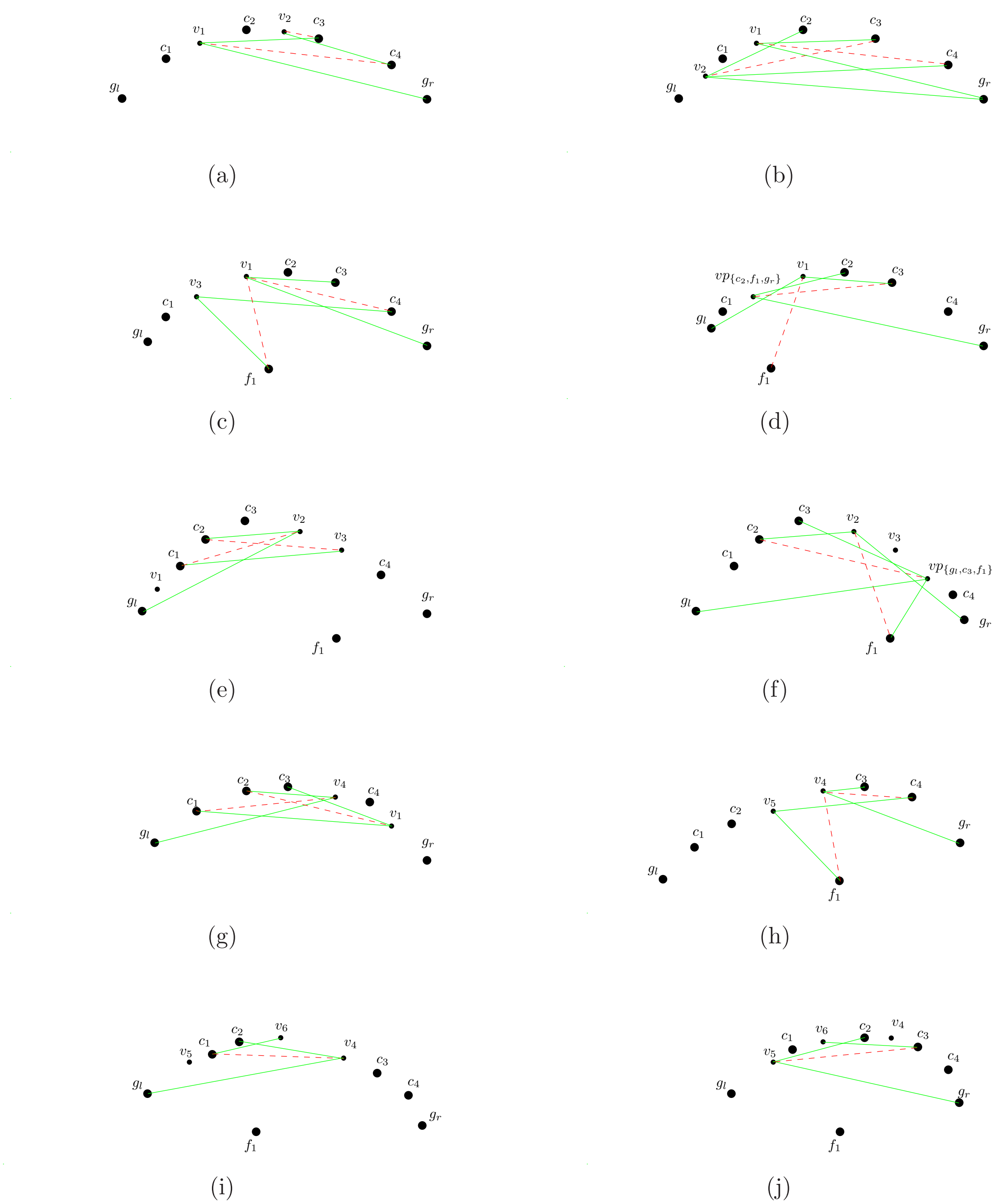


Figure 7: An illustration of Case 2.

and c_2 also by Corollary 6 as there would be points that see f_i both to the left and right of $v_{2,j}$. It cannot be on the floor between c_1 and c_2 or c_1 would not see p by Lemma 3. And finally $v_{2,j}$ and $v_{1,i}$ cannot both be to the left of c_1 (see Figure 8 (h)), as Lemma 5 with $a = g_r, b = c_2, c = c_1, p_1 = v_{1,i}$, and $p_2 = v_{2,j}$ would imply that $v_{2,j}$ must be to the left of $v_{1,i}$.

So now we suppose that $v_{2,j}$ is to the right of c_2. Applying Lemma 5 with $a = g_\ell, b = c_1, c = c_2, p_1 = v_{2,j}$, and $p_2 = p$, we have that p must be to the right of $v_{2,j}$ and that $v_{2,j}$ must be on the ceiling. But this contradicts Corollary 6 as $v_{2,j}$ would have a point that sees f_i to its left ($v_{2,i}$) and another to its right (p) (see Figure 8 (i)).

$v_{3,i}$ and $v_{3,j}$ are both on the ceiling. Note that both $v_{3,i}$ and $v_{3,j}$ must be to the right of $v_{2,i}$ or c_3 could not see them by Lemma 3. Further note that both $v_{3,i}$ and $v_{3,j}$ cannot be between c_2 and $v_{2,i}$ or c_3 would see $v_{2,i}$ by Lemma 2. So we will assume that both $v_{3,i}$ and $v_{3,j}$

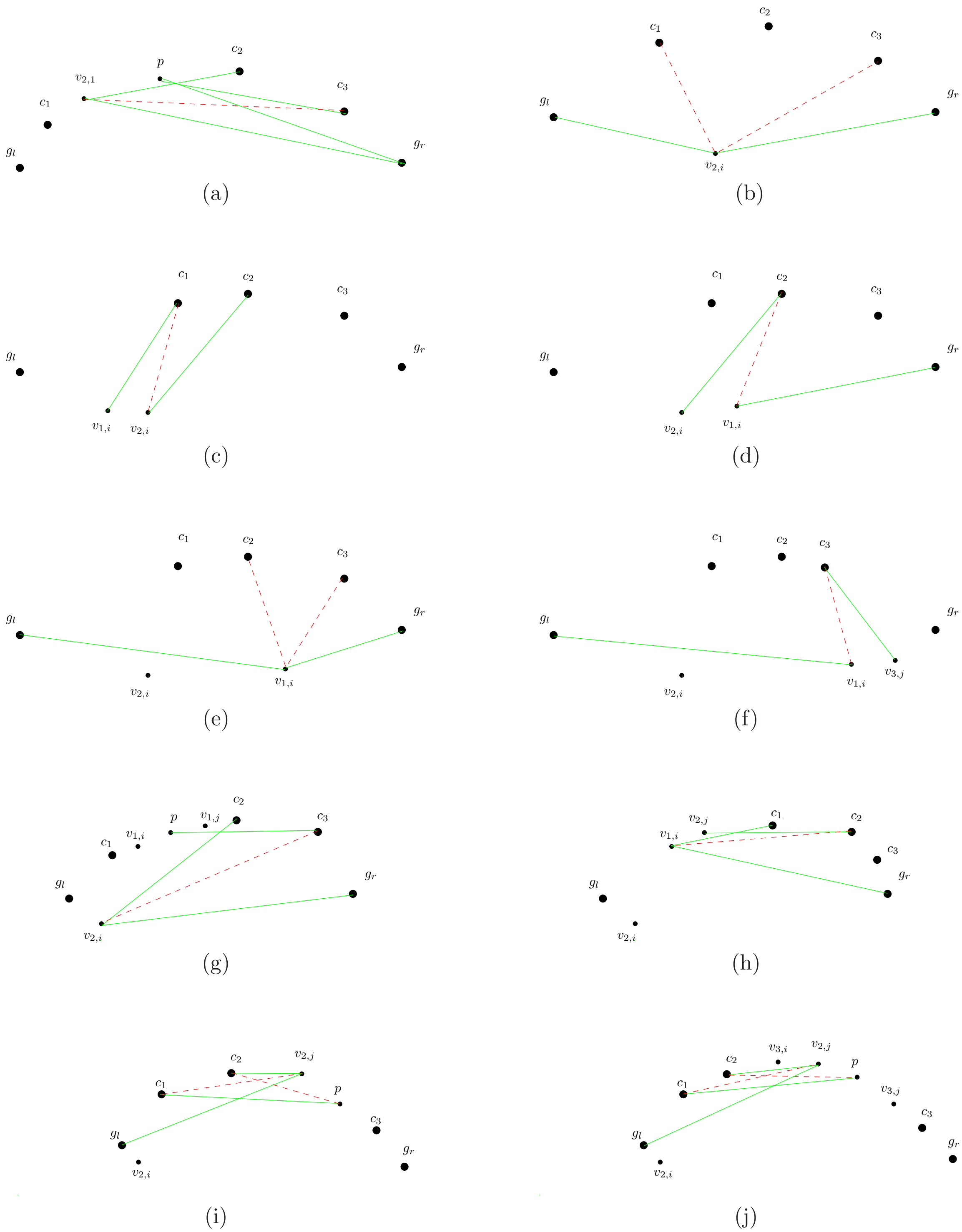


Figure 8: An illustration of Case 3.

are to the right of c_2.

Applying Corollary 6, we have that $v_{3,i}$ must be to the left of $v_{3,j}$, there can be no viewpoint that sees f_i to the right of $v_{3,j}$, and there can be no viewpoint that sees f_j to the left of $v_{3,i}$. Now consider the placement of $v_{2,j}$. Since it sees f_j, it cannot go to the left of $v_{3,i}$ so suppose we place it to the right of $v_{3,i}$. Now consider the placement of viewpoint p. Since it sees f_j, it must be placed to the right of $v_{3,i}$ and therefore is to the right of c_2. Applying Lemma 5 with $a = g_\ell, b = c_1, c = c_2, p_1 = v_{2,j}$, and $p_2 = p$, we have that p must be to the right of $v_{2,j}$ and $v_{2,j}$ must be on the ceiling. But this contradicts Corollary 6 as $v_{2,j}$ would have a point that sees f_i to its left ($v_{2,i}$) and another to its right (p) (see Figure 8 (j)).

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