Drawing the Horton Set in an Integer Grid of Minimum Size

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Abstract

In this paper we show that the Horton set of n points can be realized with integer coordinates of absolute value at most $\frac{1}{2}\left(n^{\frac{\log(n)-1}{2}}\right)$. We also show that any set of points with integer coordinates that has the same order type as the Horton set contains a point with a coordinate of absolute value at least $\left(\frac{1}{2}n\right)^{\frac{\log(n)-1}{32}}$.

1 Introduction

Throughout this paper all point sets are in general position and all logarithms are base 2. Let S be a set of n points in the plane. A *drawing* of S is a set of n points in the plane with integer coordinates and the same order type as S. For computational purposes, having integer-valued coordinates has various advantages over real-valued coordinates. For example, many combinatorial questions depend only on the order type of the point set, which is defined by the orientation of every ordered triple of points. Deciding the orientation of a triple can be done with a determinant. If the point set has integer coordinates, any possible rounding errors in this evaluation are avoided with arbitrary precision integer arithmetic. As the computational cost of these operations grows with the size of the integers, it is natural to seek drawings in which the largest absolute value of the coordinates is minimized. Moreover, large drawings require a large number of bits to be stored.

We define the *size* of a drawing as the maximum of the absolute value of its coordinates. Goodman et al. [10] found sets of n points whose smallest drawings have size $2^{2^{c_1 n}}$, and proved that every point set has a drawing with size at most $2^{2^{c_2 n}}$ (where c_1 and c_2 are constants). Our main purpose for searching for small drawings of specific classes of point sets is to have fast algorithms to generate drawings of these points sets. Afterwards, many combinatorial parameters on these point sets can be computed swiftly. Recently Bereg et al. [4] provided a linear time algorithm to generate a drawing of size $O(n^{3/2})$ of the Double Circle of $2n$ points. They also showed a lower bound of $\Omega(n^{3/2})$ on the size of every drawing of the Double Circle.

In this paper we show a drawing (that can easily be constructed in linear time) of the Horton set of size $\frac{1}{2}\left(n^{\frac{\log(n)-1}{2}}\right)$. We provide a lower bound of $\left(\frac{1}{2}n\right)^{\frac{\log(n)-1}{32}}$ on the minimum size of any drawing of the Horton set. As a corollary $\Theta(n(\log n)^2)$ bits are necessary and sufficient to store a drawing of the Horton set. In Section 2 we define and provide background on the Horton set. The upper and lower bounds are given in Sections 3 and 4 respectively.

2 The Horton Set(s)

The Horton set was introduced to give a partial solution to a problem posed by Erdős [7] in 1978. He asked whether every sufficiently large set of n points in the plane contains the vertices of a convex k-gon with no other points of the set in its interior (we call it an empty k-gon). Shortly after, Harborth [11] showed that every set of 10 points contains an empty pentagon. The case for triangles is trivial and the case for four-gons was settled in affirmative in another context by Esther Klein long time before Erdős posed this question (see [8]). Horton [12] constructed arbitrarily large point sets with no empty heptagons (and thus no larger empty k-gons). His construction is now known as the Horton set. The remaining case of empty hexagons stayed open for almost 30 years, until Nicolás [15], and independently Gerken [9], showed that every sufficiently large point set contains an empty hexagon.

Since its introduction, the Horton set has been used as an extremal example in various similar combinatorial problems on point sets. For example, as every sufficiently large set of points contains an empty k-gon for $k = 3, 4, 5, 6$, a natural question is to ask: What is the minimum number of empty k-gons in every set of n points in the plane? The case of empty triangles was first considered by Katchalski and Meir [13]; they constructed a point set with $200n^2$ empty triangles. This bound was later improved by Bárány and Füredi [2] who showed that the Horton set has $2n^2$ empty triangles. The Horton set was then used in a series of papers as a building block to construct sets with fewer empty k-gons. The first construction was given by Valtr [18], it was later improved by Dumitrescu [6], and the final improvement was given by Bárány and Valtr [3]. Devillers et al. [5] considered chromatic versions of these prob-

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lems. In particular, they described a three-coloring of the points of the Horton set with no empty monochromatic triangles. Since every set of 10 or more points contains an empty pentagon, every two-colored set of at least 10 points contains an empty monochromatic triangle. The first non-trivial lower bound (of $\Omega(n^{5/4})$) on the number of empty monochromatic triangles on every two-colored set of n points was given by Aichholzer et al [1]. This was later improved by Pach and Tóth [16] to $\Omega(n^{4/3})$. The known set with the least number of empty monochromatic triangles is given in [1] and it is based on the Horton set.

We now define the Horton set. Let S be a set of n points in general position in the plane. Sort its members by lexicographic order (first by x-coordinate and then by y-coordinate) so that $S := \{p_0, p_1, \dots, p_{n-1}\}$. Let S_{even} be the subset of its even-indexed points, and S_{odd} be the subset of its odd-indexed points. That is $S_{\text{even}} = \{p_0, p_2, \dots\}$ and $S_{\text{odd}} = \{p_1, p_3, \dots\}$.

Let X and Y be two sets of points in the plane. We say that X is *high above* Y if:

- Every line determined by two points in X is above Y.
- Every line determined by two points in Y is below X.

Definition 1 ***(Horton set)*** *A Horton set is a set H^k of 2^k points, defined recursively as:*

(1) H^0 is a Horton set.

(2) For $k > 1$, both H^k_{even} and H^k_{odd} are Horton sets.

(3) For $k > 1$ H^k_{odd} is high above H^k_{even}.

Note that a drawing of the Horton set does not necessarily satisfy Definition 1. The Horton set was described in [12] in a concrete manner. Our definition is similar to the more abstract one found in [14] (page 36), and has the advantage of supplying the structure needed for our proofs.

At this point, we should mention that in the more recent definitions of the Horton set (like the one in [14]), either H^k_{even} is high above H^k_{odd} or H^k_{odd} is high above H^k_{even}, and this relationship is allowed to change at each step of the recursion. As a result, for a fixed value of k, one gets a family of "Horton sets" (with different order types), rather than a single Horton set. Normally, this does not affect the properties that make Horton sets interesting. For example, none of the them has empty heptagons. However, in some circumstances it does, as is the case in the constructions with few empty k-gons [3]. In our case, we had to fix one of these two options in order to make the proof of our lower bound more readable. We conjecture that our hold for the general setting. Another difference with the definitions found in the literature is that no two points are allowed to have the same x-coordinate. So usually the points of H are sorted by their x-coordinate rather than lexicographically. Because we are trying to bound the size of any drawing of the Horton set, we need to relax this condition a little.

To show that Horton sets do exist, let:

- $H^0 = \{(1,1)\}$.
- $H^1 = \{(1,1),(2,2)\}$.
- $H^k = \{(2x-1, y) : (x,y) \in H^{k-1}\} \cup \{(2x, y+d_{k-1}) : (x,y) \in H^{k-1}\}$ for $k \geq 2$.

  where $d_k := 3^{2^k}$.

This drawing of the Horton set is given in [2]. Note that it has size at least 3^n. All other drawings we have seen in the literature are also of exponential size. Then again, to the best of our knowledge nobody has ever tried to find drawings of small size.

3 Upper bound

In this section we prove our upper bound by constructing a drawing P^k of the Horton set H^k of $n = 2^k$ points. Let:

$$f(k) = \begin{cases} 0 & \text{if } k = 1 \\ 2^{\frac{k(k-1)}{2}-1} & \text{if } k \geq 2 \end{cases}$$

$$g(k) = \begin{cases} 0 & \text{if } k = 1 \\ f(k) - f(k-1) & \text{if } k \geq 2 \end{cases}$$

We use f and g to construct P^k recursively. Let $P^0 := \{(0,0)\}$. For $k \geq 1$, let $P^k_{\text{even}} := \{(2x, y) : (x,y) \in P^{k-1}\}$, $P^k_{\text{odd}} := \{(2x+1, y+g(k)) : (x,y) \in P^{k-1}\}$ and $P^k := P^k_{\text{even}} \cup P^k_{\text{odd}}$. For $k > 1$, the largest x-coordinate of P^k is $n-1$, and its largest y-coordinate is $\sum_{i=1}^k g(k) = f(k) = 2^{\frac{k(k-1)}{2}-1} = \frac{1}{2}\left(n^{\frac{\log(n)-1}{2}}\right)$. Therefore, P^k has size $\frac{1}{2}\left(n^{\frac{\log(n)-1}{2}}\right)$.

Theorem 1 *There is a drawing of the Horton set of $n = 2^k$ points of size $\frac{1}{2}\left(n^{\frac{\log(n)-1}{2}}\right)$.*

Proof. It only remains to show that P^k is a Horton set. By definition P^0 and P^1 are Horton sets. By induction, assume that $k \geq 2$, and that P^k_{even} and P^k_{odd} are Horton sets. It remains to show that P^k_{odd} is high above P^k_{even}. Let $p_0, p_1, \dots, p_{2^k}$ be the points of P^k in lexicographical order. The largest y-coordinate of P^k_{even} is $f(k-1)$ and the smallest y-coordinate of P^k_{odd} is $g(k) = f(k) - f(k-1)$. Let $0 \leq i < j \leq n$ be two even integers and let ℓ be the directed line from p_i to p_j. We show that every point of P^k_{odd} is above ℓ, or rather that every point of P^k_{odd} is to the left of ℓ. By induction P^k_{odd} is above the line segment joining p_1 and p_{n-1} and these points are above $(1, g(k))$ and $(n-1, g(k))$. Therefore, it suffices to show that both $(1, g(k))$ and $(n-1, g(k))$ are to the left of ℓ.

The slope of ℓ is at least $-f(k-1)/2$. So if $(1, g(k))$ is to the left of the directed line from $(n-6, f(k-1))$ to $(n-4, 0)$, then it is also to the left of ℓ. This is the case since:

$$\begin{aligned}
&\begin{vmatrix} n-6 & f(k-1) & 1 \\ n-4 & 0 & 1 \\ 1 & g(k) & 1 \end{vmatrix} \\
= \quad & -g(k)(-2) + f(k-1) - (n-4)f(k-1) \\
= \quad & 2f(k) - (n-3)f(k-1) \\
= \quad & 2^{\binom{k-1}{2}}\left[2^k - (2^k - 3)\right] \\
> \quad & 0
\end{aligned}$$

Finally, the slope of ℓ is at most $f(k-1)/2$. So if $(n-1, g(k))$ is to the left of the directed line from $(0,0)$ to $(2, f(k-1))$, then it is also to the left of ℓ. Again, this is the case since:

$$\begin{aligned}
&\begin{vmatrix} 0 & 0 & 1 \\ 2 & f(k-1) & 1 \\ n-3 & g(k) & 1 \end{vmatrix} \\
= \quad & 2(f(k) - f(k-1)) - (n-3)f(k-1) \\
= \quad & 2f(k) - (n-1)f(k-1) \\
= \quad & 2^{\binom{k-1}{2}+k} - (2^k - 1)2^{\binom{k-1}{2}-1} \\
= \quad & 2^{\binom{k-1}{2}-1}\left[2^{k+1} - 2^k + 1\right] \\
> \quad & 0.
\end{aligned}$$

An analogous proof shows that every line through two points of P^k_{odd} is above every point of P^k_{even}. This completes the proof. □

4 Lower bound

The proof of the lower bound on the size of any drawing of the Horton set is more technical and requires some notation before proceeding.

As mentioned before, a drawing of the Horton set might not satisfy Definition 1. We call a drawing that does an *isothetic* drawing of the Horton set. We first show a lower bound on the size of an isothetic drawing of the Horton set; afterwards we consider the general case.

Throughout this section, let P be an isothetic drawing of the Horton set of $n = 2^k$ points. Let $p_0, p_1, \ldots, p_{n-1}$ be the members of P in lexicographical order. Define recursively a rooted binary tree, T, on subsets of P as follows: P is at the root; if Q is a vertex of T of at least two points, then Q_{even} and Q_{odd} are its left and right children respectively. By construction, the vertices of T are sets of 2^i points of P for some $0 \leq i \leq k$. Let T_i be the vertices of T that consist of exactly 2^i points of P. We call T_i the i-th level of T. The following properties of T_i are easily verified: the vertices of T_i are precisely the subsets of points of P whose indices are congruent modulo 2^{k-i}; every vertex in T_i is at distance $k-i$ from the root and between the leftmost and rightmost vertices of T_i, there are $2^k - 2^{k-i} + 1$ points of P.

For $0 < t \leq k$, a *t-vertical partition* is a partition of the $n/2$ middle points $p_{2^{k-2}}, p_{2^{k-2}+1}, \ldots, p_{3 \cdot 2^{k-2}-1}$ of P into sets of equal size by $2^{t-1}+1$ vertical lines. Specifically it is a set $\ell_0, \ell_1 \ldots \ell_{2^{t-1}}$ of vertical lines, such that l_i is between the points $p_{2^{k-2}+2^{k-t}i-1}$ and $p_{2^{k-2}+2^{k-t}i}$. Therefore, between l_i and l_{i+1} there are exactly 2^{k-t} points of P and exactly 2^{l-t} points of every subset of P in l-th level of T, for $l > t$.

Lemma 2 *Let R be the region bounded by ℓ_0 and $\ell_{2^{t-1}}$ in a vertical t-partition of P. Let Q_1 and Q_2 be two subsets of P which are vertices of T_1. If γ_1 and γ_2 are the supporting lines of Q_1 and Q_2 respectively, then they do not intersect inside R.*

Proof. We prove it for when Q_1 is a sibling of Q_2, the general case follows easily. Without loss of generality assume that Q_1 is a left child and Q_2 is a right child. Let q_{l1} and q_{l2} be the leftmost points of Q_1 and Q_2 respectively. Likewise, let q_{r1} and q_{r2} be the rightmost points of Q_1 and Q_2 respectively.

For every vertex of T label the edge incident to its left child with "0" and the edge incident to its right child with "1". Note that by construction of T, the binary expansion of an index i is precisely the labels of the edges in the unique path from the root to p_i in T. The last two labels in the path from the root to left child of Q_1 are "00" and the the last two labels form the root to the right child of Q_2 are "11". Therefore q_{l1} is to the left of ℓ_0 and q_{r2} is to the right of $\ell_{2^{t-1}}$. Note that there are $2^{k-1}+1$ points of P between q_{l1} and q_{r1}, and $2^{k-1}+1$ points of P between q_{l2} and q_{r2}. Therefore, neither Q_1 and Q_2 can be contained in R, and both q_{r1} and q_{l2} are inside R.

Since the convex hulls of Q_1 and Q_2 are disjoint, γ_1 and γ_2 cannot intersect inside R. Otherwise either γ_1 is above q_{l2} or γ_2 is below q_{r1}, a contradiction. □

In what follows, fix a vertical t-partition of P. An immediate consequence of Lemma 2 is that in R, there is a bottom-up order of the lines defined by every subset of P at the first level of T. This order coincides with the left to right order in T_1. In fact every subset of P that is a vertex of T with more than two points is the union of its descendants in T_1. Therefore we can extend this order to every level of T.

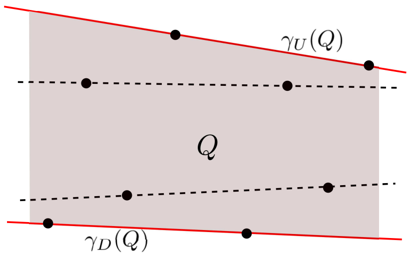

Figure 1: The bounding lines of Q.

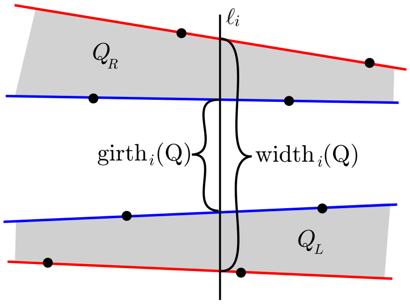

Figure 2: The width and girth of Q with respect to ℓ_i.

Let Q be a vertex of T with more than 2 two points and let $P(Q)$ be its parent. If Q is the left child of $P(Q)$, let $S(Q)$ be the right child of Q; otherwise let $S(Q)$ be the left child of Q. Let $\gamma_D(Q)$ be the line containing the leftmost descendant of Q. Let $\gamma_U(Q)$ be the line containing the rightmost descendant of Q. Note that Q is bounded from below by $\gamma_D(Q)$ and from above $\gamma_U(Q)$; see Figure 1.

Let ℓ_i be a line of the t-vertical partition. We define the *width*, $\text{width}_i(Q)$, of Q with respect to ℓ_i as the distance between the intersection points of $\gamma_D(Q)$ and $\gamma_U(Q)$ with ℓ_i. Let Q_L and Q_R be the left and right children of Q respectively. We define the *girth*, $\text{girth}_i(Q)$, of Q with respect to ℓ_i as the distance between the intersection points of $\gamma_U(Q_L)$ and $\gamma_D(Q_R)$ with ℓ_i; see Figure 2.

Our general approach is to lower bound the girth of a vertex of T in terms of the girth of one of its children. This bound is expressed in the following lemma.

Lemma 3 *Let ℓ_i and ℓ_j $(j > i+1)$ be two lines of the vertical t-partition. Let Q be a vertex of T_l $(t < l < k)$. If the distance between ℓ_i and ℓ_{i+1} is d_1, and the distance between ℓ_{j-1} and ℓ_j is d_2, then:*

(1) $\text{girth}_i(P(Q)) \geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{l-t-1}\,\text{girth}_j(Q) - \text{width}_i(S(Q))$

(2) $\text{girth}_j(P(Q)) \geq \left(\frac{(d_2)^2}{(d_1+d_2)d_1}\right) 2^{l-t-1}\,\text{girth}_i(Q) - \text{width}_j(S(Q))$

Proof. We will prove inequality (1); the proof of (2) is analogous. Assume that Q is the left child $P(Q)$ and let Q' be the right child of $P(Q)$. The case when Q is the right child of $P(Q)$ can be proven with similar arguments. Note that by our assumption on Q, $Q_R = S(Q)$.

Let p_1' and p_2' be two consecutive points in Q_L between ℓ_{j-1} and ℓ_j at a distance at most $d_2/2^{l-t-1}$ from each other. Such a pair exists as there are 2^{l-t-1} points of Q_L between ℓ_{j-1} and ℓ_j.

Let p'' be the point between them in Q_R. Let φ be the line through p_2' and p''. Note that the slope of φ with respect to $\gamma_D(Q_R)$ is at most $-\min\left\{\text{girth}_{j-1}(Q), \text{girth}_j(Q)\right\} \cdot 2^{l-t-1}/d_2$. Since trivially $\text{girth}_{j-1}(Q) \geq \frac{d_1}{d_1+d_2}\text{girth}_j(Q)$, this is at most $-\frac{d_1}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_j(Q)$. Let $q_1 := \gamma_D(Q_R) \cap \ell_i$, $q_2 := \varphi \cap \ell_i$ and $q_3 := \gamma_D(Q') \cap \ell_i$.

Since there is at least a point in $\gamma_D(Q') \cap P$ to the left of ℓ_i and above φ, q_2 cannot be above q_3. Therefore the distance from q_1 to q_2 is at most the distance from q_1 to q_3. Note that the distance from q_1 to q_3 is precisely $\text{girth}_i(P(Q)) + \text{width}_i(S(Q))$. We now show that the distance from q_1 to q_2 is at least $\frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_j(Q)$—this completes the proof of (1).

Let φ' be the line parallel to φ and passing through the intersection point of ℓ_{j-1} and $\gamma_D(Q_R)$. Note that φ' is below φ. Therefore, the distance from q_1 to the intersection point of φ' and ℓ_i is at most the distance from q_1 to q_2. But this first distance is at least $\frac{(d_1)^2}{(d_1+d_2)d_2}2^{l-t-1}\,\text{girth}_j(Q)$. □

Two obstacles may prevent us from directly applying Lemma 3. One is that the difference between d_1 and d_2 may be too big and in consequence $\frac{(d_1)^2}{(d_1+d_2)d_2}$ or $\frac{(d_2)^2}{(d_1+d_2)d_1}$ is too small. This can be easily fixed by taking an appropriate value for t and then choosing appropriate values for i and j. We do this in Lemma 4. The other problem is that the second term in the right hand sides of inequalities (1) and (2) of Lemma 3 may be too large. In this case, we need to prune T to get rid of vertices of large width. We show how to do this in Lemma 5.

Lemma 4 *If $t = \lceil \log k^2 \rceil$ and $k \geq 16$, then either P has size $n^{\frac{1}{2}\log n}$ or there are two indices $j > i+1$ such that the ratio of the distance between ℓ_i and ℓ_{i+1} and the distance between ℓ_{j-1} and ℓ_j is at least $1/2$ and at most 2.*

Proof. Let $d_1' \leq d_2' \leq \cdots \leq d_{2^{t-1}}'$ be the distances between two consecutive lines of the vertical partition. We look for a pair such that one is at most two times the other. Suppose there is no such pair; then $d_{i+1}' \geq 2d_i'$. Since between the two lines defining d_1 there are exactly 2^{k-t} points of P, and no three of them have the same

x-coordinate, $d_1 \geq 2^{k-t-1}$. Therefore:

$$d_{2^t-2} \geq 2^{k-t-1} \cdot 2^{2^{t-1}-1} \geq 2^{\frac{1}{2}k^2+k-t-2} \geq n^{\frac{1}{2}\log n}$$

The latter part of the inequality follows from our assumption that $k \geq 16$ □

Lemma 5 *For $0 \leq l \leq k-1$, let $Q_1, Q_2, \ldots, Q_{2^{k-l}}$ be the vertices of T_l in their left to right order. Let P' be the set that results from removing from P the points that lie in a Q_i with an even (or odd) index and let T' be its corresponding tree. For every vertex Q' in T', let Q be the smallest vertex of T that contains Q'. Then, P' is an isotethic drawing of the Horton set and $S(Q') \subset S(Q)$ for every vertex Q' in the $(l+1)$-th level or higher of T'.*

Proof. Assume that the even-indexed Q_i's where removed, the other case is analogous. Let $s = k - l$. If $s = 1$, then P' is trivially an isothetic drawing of the Horton set on $n/2$ points, since it is equal to P_{odd}. Suppose that $s > 1$. Then, P_{even} and P_{odd} are each isothethic drawings of the Horton set of $n/2$ points. Each Q_i that was removed is either contained entirely in P_{even} or in P_{odd}. So by induction removing these Q_i from P_{even} and P_{odd} provides isothetic drawings of the Horton set on $n/4$ points. Let P''_0 and P''_1 be these sets, respectively. We claim that P' is constructed from P by alternatively removing and keeping intervals of 2^{k-l-1} consecutive points of P, so that $p_0, \ldots, p_{2^{k-l-1}-1}$ are removed, $p_{2^{k-l-1}}, \ldots, p_{2^{k-l}-1}$ are kept and so on. For $s = 1$, this is trivial since $P' = P_{\text{odd}}$. For $s > 1$, P''_0 and P''_1 are constructed from P_{even} and P_{odd} by removing and keeping intervals of 2^{k-l-2} consecutive points of P_{even} and P_{odd}, respectively. Each interval of 2^{k-l-2} points that was removed from P_{even}, together with an interval of 2^{k-l-2} that was removed from P_{odd}, forms and interval of 2^{k-l-1} that is removed from P. The same holds for the intervals of P_{even} and P_{odd} that are kept. Therefore $P'_{\text{even}} = P''_0$ and $P'_{\text{odd}} = P''_1$. Since $P'_{\text{even}} \subset P_{\text{even}}$ and $P'_{\text{odd}} \subset P_{\text{odd}}$, P'_{odd} is high above P'_{even}, and P' is a Horton set. The later part of the Lemma follows easily from the previous observations. □

Theorem 6 *For a sufficiently large value of k, every isothetic drawing of the Horton set of $n = 2^k$ points has size at least $n^{\frac{1}{8}\log n}$.*

Proof. Let $t = \lceil \log k^2 \rceil$, and let d_i be the distance between ℓ_i and ℓ_{i+1}. By Lemma 4 we may assume then that there exists a pair of indices $j > i+1$ such that the ratio between d_i and d_j is at least $1/2$ and at most 2. Without loss of generality suppose that $d_j \geq d_i$. Let $D := \sum_{i=1}^{2^t-2} d_i$. We may assume that $D < n^{\frac{1}{8}\log n}$ as otherwise we are done.

Let Q be any vertex in the $(t+1)$-th level of T. Note that there are exactly two points of Q between ℓ_i and ℓ_{i+1}, and exactly two points of Q between ℓ_{j-1} and ℓ_j. Since these four points have integer coordinates, by Pick's theorem [17] the area of their convex hull is at least one. Therefore so is the area of the trapezoid bounded by $\gamma_D(Q)$, $\gamma_U(Q)$, ℓ_i and ℓ_j. But this area is at most $D(\text{width}_i(Q) + \text{width}_j(Q)/2)$. Therefore $\max\{\text{width}_i(Q), \text{width}_j(Q)\} \geq 1/D$. This bound also holds for every vertex at a level higher than $t+1$.

Let l be the largest positive integer $t < l \leq k$ such that there exists a vertex R in the l-th level of T that satisfies:

$$\max\{\text{width}_i(S(R)), \text{width}_j(S(R))\} \geq \frac{2^{(l-t-6)(l-t-7)/2}}{D} \tag{1}$$

Such an l and R exist since (1) holds for every vertex at the $(t+6)$-th level of T. We may assume that $l < k$, otherwise P has size at least $n^{\frac{1}{8}\log n}$ (for a sufficiently large value of k). Remove all the vertices in the l-th level of T, with the same parity as R, as in Lemma 5. By the second part of Lemma 5 no vertex of T' in a level higher than l satisfies (1). Without loss of generality assume that $\text{width}_i(S(R)) \geq (2^{(l-t-6)(l-t-7)/2})/D$ Let $(P(R)' = Q'_l, Q'_{l+1}, \ldots, Q'_{k-1} = P')$ be the path from $P(R)'$ to the root of T'. We will prove inductively for $l \leq m \leq k-1$, that:

$$\text{girth}_i(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \equiv l \mod 2 \tag{2}$$

$$\text{girth}_j(Q'_m) \geq \frac{2^{(m-t-6)(m-t-7)/2}}{D} \text{ if } m \not\equiv l \mod 2 \tag{3}$$

This holds for $m = l$ since $\text{girth}_i(Q'_{l+1}) = \text{girth}_i(P(R)') \geq \text{width}_i(S(R)) \geq (2^{(l-t-6)(l-t-7)/2})/D$ Assume then that $m > l$ and that it holds for smaller values of m. Suppose that m has the same parity as l. Then by inequality (1) of Lemma 3:

$$\begin{aligned}
\text{girth}_i(Q'_m) &\geq \left(\frac{(d_1)^2}{(d_1+d_2)d_2}\right) 2^{m-t-2}\,\text{girth}_j(Q'_{m-1}) \\
&\quad - \text{width}_i(S(Q'_{m-1})) \\
&\geq 2^{m-t-5}\,\text{girth}_j(Q'_{m-1}) \\
&\quad - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-5}\,\frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\quad - \frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&\geq 2^{m-t-6}\,\frac{2^{(m-t-7)(m-t-8)/2}}{D} \\
&= \frac{2^{(m-t-6)(m-t-7)/2}}{D}
\end{aligned}$$

Therefore P has size at least $n^{\frac{1}{8}\log n}$, for a sufficiently large value of n. The proof when m has different parity as l is similar, but uses inequality (2) of Lemma 3 instead. □

We are now ready to prove the general bound.

Theorem 7 *Every drawing of the Horton set of* $n = 2^k$ *points has size* $\left(\frac{1}{2}n\right)^{\frac{\log(n)-1}{32}}$*, for a sufficiently large value of* n*.*

Proof. Let P' be a (not necessarily isotethic) drawing of the Horton set of n points. As P and P' have the same order type we can label P' with the same labels as P, such that corresponding triples of points in P and P' have the same orientation. Let then $\{p'_0, \dots, p'_{n-1}\}$ be P' with these labels.

Note that the clockwise order by angle around p'_0 of P'_{odd} is $(p'_1, p'_2, \dots)$, and that p'_0 lies in an unbounded cell of the line arrangement of the lines defined by every pair of points of P'_{odd}. We may move p'_0 towards infinity without changing this radial order around p'_0. Therefore there is a direction $\vec{d}$ in which we can project orthogonally P'_{odd} so that the order of the projection is precisely $(p'_1, p'_2, \dots)$. Rotate $\vec{d}$ until it coincides with a direction defined by a pair of points of P'. Let $v = (a, b)$ be the direction vector defined by this pair. Note hat v has integer coordinates. Moreover if we project in this direction instead the order of P'_{odd} does not change. We may assume that $||v|| \leq (n/2)^{\frac{1}{32}\log(n/2)}$ as otherwise we are done. Let $v^{\perp} = (b, -a)$. Consider a change of basis from the standard basis to $\{v, v^{\perp}\}$. Note that under this transformation (x, y) is mapped to $\left(\frac{ax+by}{a^2+b^2}, \frac{ay-bx}{a^2+b^2}\right)$. If we multiply the image of P' under this mapping by $a^2 + b^2$ we obtain an isothethic drawing of the Horton set on $n/2$ points. By Theorem 6, this drawing has size at least $(n/2)^{\frac{1}{8}\log(n/2)}$.Therefore, P' has size at least $((n/2)^{\frac{1}{8}\log(n/2)})/(a^2 + b^2) \geq (n/2)^{\frac{\log(n)-1}{16}}$. □

Finally we point out that the constants in the exponent of the lower bounds of Theorems 6 and 7 can be improved. We preferred to simplify the exposition at the expense of these worse bounds.

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