On the Spanning Ratio of Constrained Yao-Graphs

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Abstract

We present upper bounds on the spanning ratio of constrained Yao-graphs with at least 7 cones. Given a set of points in the plane, a Yao-graph partitions the plane around each vertex into k disjoint cones, each having aperture $\theta = 2\pi/k$, and adds an edge to the closest vertex in each cone. Constrained Yao-graphs have the additional property that no edge properly intersects any of the given line segment constraints. We show that constrained Yao-graphs with an even number of cones ($k \geq 8$) have spanning ratio at most $1/\left(1 - 2\sin(\theta/2)\right)$ and constrained Yao-graphs with an odd number of cones ($k \geq 7$) have spanning ratio at most $1/\left(1 - 2\sin(3\theta/8)\right)$. These bounds match the current upper bounds in the unconstrained setting.

1 Introduction

A geometric graph G is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. Every edge is weighted by the Euclidean distance between its endpoints. The distance between two vertices u and v in G, denoted by $d_G(u, v)$, is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for $t \geq 1$) if for each pair of vertices u and v, $d_H(u, v) \leq t \cdot d_G(u, v)$. The smallest value t for which H is a t-spanner is the *spanning ratio* or *stretch factor*. The graph G is referred to as the *underlying graph* of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [8, 15] for a comprehensive overview of the topic). We look at a specific type of geometric spanner: Yao-graphs.

Introduced independently by Flinchbaugh and Jones [14] and Yao [16], Yao-graphs partition the plane around each vertex into k disjoint cones, each having aperture $\theta = 2\pi/k$. The Y_k-graph is constructed by, for each cone of each vertex u, connecting u to the vertex v that is closest to u. However, neither Flinchbaugh and Jones nor Yao proved that these graphs are spanners. To the best of our knowledge, the first such proof was given by Althöfer et al. [1], who proved that for every spanning ratio $t > 1$, there exists a k such that the Y_k-graph is a t-spanner. It appears that a similar result was already known by that time, since Clarkson [10] remarked in 1987 that the Y_{12}-graph is a $1 + \sqrt{3}$-spanner, though without providing a proof or reference.

In 2004, Bose et al. [7] provided a more precise bound on the spanning ratio. They showed that Yao-graphs with at least 9 cones have spanning ratio at most $1/(\cos\theta - \sin\theta)$. This was later strengthened to show that Yao-graphs with at least 7 cones are $1/\left(1 - 2\sin(\theta/2)\right)$-spanners [3]. Recently, Damian and Raudonis [11] showed that the Y_6-graph is a 17.64-spanner and Bose et al. [4] showed that the Y_4-graph has spanning ratio at most 663. Barba et al. [2] showed that the Y_5-graph is a $\left(2 + \sqrt{3}\right)$-spanner. In the same paper, they also improved the upper bound on the spanning ratio of the Y_6-graph to 5.8 and that of Yao-graphs with an odd number of cones to $1/\left(1 - 2\sin(3\theta/8)\right)$. On the other hand, when a Yao-graph has less than 4 cones, El Molla [13] showed that there is no constant t such that it is a t-spanner.

The above results, however, focus on Yao-graphs where the underlying graph is the complete Euclidean geometric graph. We study this problem in a more general setting with the introduction of line segment *constraints*. Specifically, let P be a set of points in the plane and let S be a set of line segments between two vertices in P, called *constraints*. The set of constraints is plane, i.e. no two constraints intersect properly. A vertex in P can be the endpoint of multiple constraints. Two vertices u and v can see each other if and only if either the line segment uv does not properly intersect any constraint or uv is itself a constraint. If two vertices u and v can see each other, the line segment uv is a *visibility edge*. The *visibility graph* of P with respect to a set of constraints S, denoted $Vis(P, S)$, has P as vertex set and all visibility edges as edge set. In other words, it is the complete graph on P minus all edges that properly intersect one or more constraints in S.

This setting has been studied extensively within the context of motion planning amid obstacles. Clarkson [10] was one of the first to study this problem and showed how to construct a linear-sized $(1 + \epsilon)$-spanner of $Vis(P, S)$. Subsequently, Das [12] showed how to construct a spanner of $Vis(P, S)$ with constant spanning ratio and constant degree. The Constrained Delaunay Triangulation was shown to be a 2.42-spanner of $Vis(P, S)$ [6]. Recently, it was also shown that the constrained θ_6-graph is a 2-spanner of $Vis(P, S)$ [5] and this spanning ratio is tight,

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i.e. there is a matching lower bound. This result was generalized to show that the constrained $\theta_{(4k+2)}$-graph ($k \geq 1$) has a tight spanning ratio of $1 + 2\sin(\theta/2)$ [9]. This paper also improved the spanning ratio of the other constrained θ-graphs with at least 6 cones.

To the best of our knowledge, Yao-graphs have not been considered in the constrained setting. As such, it is unknown whether they are spanners of $Vis(P, S)$. In this paper, we set an important first step towards answering this question by showing that constrained Yao-graphs with at least 7 cones are spanners. In particular, we prove that constrained Yao-graphs with an even number of cones have spanning ratio at most $1/(1 - 2\sin(\theta/2))$. When the constrained Yao-graph has an odd number of cones, we can improve on this result and show an upper bound of $1/(1 - 2\sin(3\theta/8))$. These bounds match the current upper bounds in the unconstrained setting.

2 Preliminaries

We define a *cone* C to be the region in the plane between two rays originating from a vertex referred to as the apex of the cone. When constructing a (constrained) Y_k-graph, for each vertex u consider the rays originating from u with the angle between consecutive rays being $\theta = 2\pi/k$. Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical halfline through u that lies above u. Let this cone be C_0 of u and number the cones in clockwise order around u (see Figure 1). The cones around the other vertices have the same orientation as the ones around u. We write C_i^u to indicate the i-th cone of a vertex u. For ease of exposition, we only consider point sets in general position: no two points lie on a line parallel to one of the rays that define the cones and no three points are collinear.

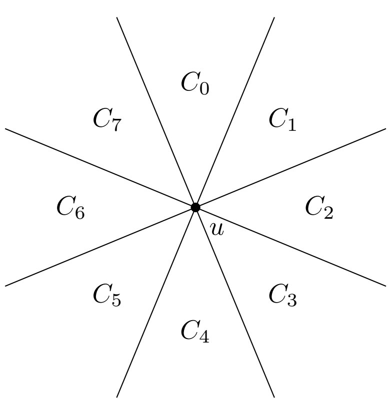

Figure 1: The cones having apex u in the Y_8-graph

Let vertex u be an endpoint of a constraint c and let the other endpoint v lie in cone C_i^u. The lines through all such constraints c split C_i^u into several *subcones*. We use $C_{i,j}^u$ to denote the j-th subcone of C_i^u (see Figure 2). When a constraint $c = (u, v)$ splits a cone of u into two subcones, we define v to lie in both of these subcones. We consider a cone that is not split to be a single subcone.

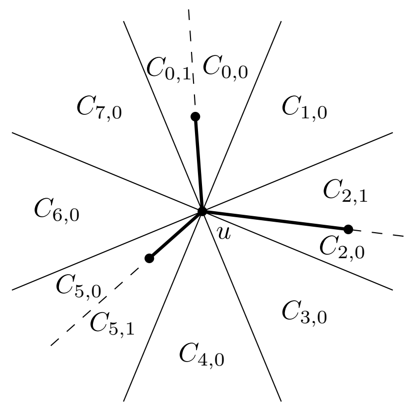

Figure 2: The subcones having apex u in the Y_8-graph. Constraints are shown as thick segments

We now introduce the constrained Y_k-graph: for each subcone $C_{i,j}$ of each vertex u, add an edge from u to the closest vertex in that subcone that can see u (see Figure 3). When there exist multiple closest vertices in a subcone, we add an edge to only a single one of them. More formally, we add an edge between two vertices u and v if v can see u, $v \in C_{i,j}^u$, and for all points $w \in C_{i,j}^u$ that can see u, $|uv| \leq |uw|$, where $|xy|$ denotes the length of the line segment between two points x and y and ties are broken arbitrarily.

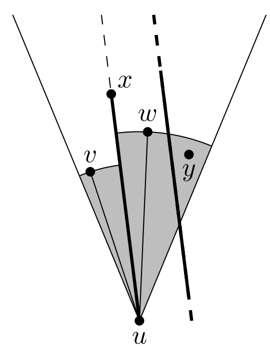

Figure 3: Vertex v is the closest visible vertex to u in the left subcone and w is the closest visible vertex to u in the right subcone, since y is not visible to u

Finally, we re-introduce a property of visibility graphs. Though the following lemma was applied to constrained θ-graphs in [5], the property holds for any visibility graph. To avoid confusion, we explicitly define that we call a region *empty* if it does not contain any vertex of P.

Lemma 1 *Let u, v, and w be three arbitrary points in the plane such that uw and vw are visibility edges and w is not the endpoint of a constraint intersecting the interior of triangle uvw. Then there exists a convex chain of visibility edges from u to v in triangle uvw, such that the polygon defined by uw, wv and the convex chain is empty and does not contain any constraints.*

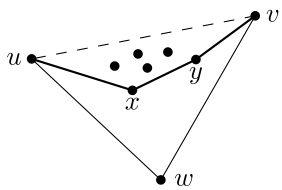

Figure 4: The convex chain between vertices u and v, where thick lines are visibility edges

3 Spanning Ratio

In this section, we prove that constrained Yao-graphs with at least 7 cones are spanners of the visibility graph.

Theorem 2 *The constrained Y_k-graph ($k \geq 7$) is a $1/\left(1 - 2\sin\left(\frac{\theta}{2}\right)\right)$-spanner of $Vis(P,S)$.*

Proof. Let u and w be two vertices that can see each other. We show that there exists a path connecting u and w in the constrained Y_k-graph ($k \geq 7$) of length at most $t \cdot |uw|$ for $t = 1/(1 - 2\sin(\theta/2))$, by induction on the distance between every pair of vertices u and w that can see each other. For ease of exposition, we assume without loss of generality that $w \in C_0^u$.

Base case: Vertices u and w are a closest visible pair. Since the closest visible pair need not be unique, we proceed to show that the subcone of C_0^u that contains w does not contain any vertices visible to u at distance at most $|uw|$: If there were such a vertex x, since ux and xw are visibility edges that lie in the same subcone, by Lemma 1 there exists a convex chain of visibility edges connecting x to w. Since we have at least 7 cones, the vertex adjacent to w along this chain is strictly closer to w than u, contradicting that $|uw|$ is a closest visible pair. Hence, since w is the closest visible vertex, uw is an edge in the constrained Y_k-graph and thus there exists a path between u and w of length $|uw| < t \cdot |uw|$.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices that can see each other and whose distance is less than $|uw|$.

If uw is an edge in the constrained Y_k-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v be the closest visible vertex to u in the subcone of u that contains w, and let x be the point along uw such that $|uv| = |ux|$ (see Figure 5). Since x lies on uw, both ux and xw are visibility edges.

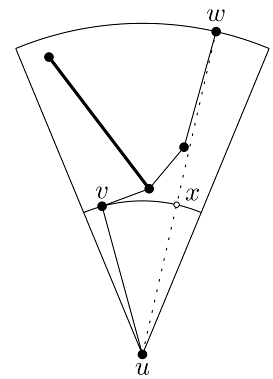

Figure 5: A convex chain from v to w

Next, we show that vx is also a visibility edge: If vx is not a visibility edge, that implies that it crosses some constraint. Since uv and ux are visibility edges, this constraint cannot cross them. Therefore, one endpoint of the constraint is contained in triangle uvx. Let y be this endpoint. Since v and w lie in the same subcone of u, u is not the endpoint of a constraint intersecting the interior of uvx. Hence, we can apply Lemma 1 and obtain a convex chain of visibility edges from v and x and the polygon defined by uv, ux, and the convex chain is empty and does not contain any constraints. This implies that u can see every vertex along the convex chain, each of which is closer to it than v, contradicting that v was the closest visible vertex to u.

Since vx and xw are visibility edges, we can apply Lemma 1 to triangle vxw and we obtain a convex chain of visibility edges $v = p_0, ..., p_j = w$ connecting v and w (see Figure 5). Since we have at least 7 cones, the distance between any consecutive pair of vertices is strictly less than $|uw|$. Hence, since every consecutive pair of vertices along this convex chain can see each other, we can apply induction on each of them. Therefore, there exists a path from u to w via v of length at most

$$|uv| + t \cdot \sum_{i=0}^{j-1} |p_i p_{i+1}|.$$

Since the chain between v and w is contained in triangle vxw and the chain is convex, it follows that the total length of the chain is at most $|vx| + |xw|$. Thus, we can upper bound the length of the path by

$$|uv| + t \cdot (|vx| + |xw|)\,.$$

Since $|uv| = |ux|$, triangle uvx is an isosceles triangle and we can express $|vx|$ as $2\sin(\angle vux/2) \cdot |uv|$. Since this function is increasing for $\angle vux \in [0, 2\pi/7]$ and $\angle vux$ is at most θ, it follows that $|vx| \leq 2\sin(\theta/2) \cdot |uv|$. Next, we look at $|xw|$: Since x lies on uw and $|uv| = |ux|$, it follows that $|xw| = |uw| - |ux| = |uw| - |uv|$. Hence, the path between u and w has length at most

$$\begin{aligned} & |uv| + t \cdot (|vx| + |xw|) \\ \leq\ & |uv| + t \cdot \left(2\sin\left(\frac{\theta}{2}\right) \cdot |uv| + |uw| - |uv| \right) \\ =\ & t \cdot |uw| + \left(1 + 2\sin\left(\frac{\theta}{2}\right) \cdot t - t \right) \cdot |uv|. \end{aligned}$$

Hence, for the length of the path to be at most $t \cdot |uw|$, we need that

$$1 + 2\sin\left(\frac{\theta}{2}\right) \cdot t - t \leq 0,$$

which can be rewritten to

$$t \geq \frac{1}{1 - 2\sin\left(\frac{\theta}{2}\right)},$$

completing the proof. □

For odd values of k, the spanning ratio can be decreased a bit: Let C_i^u be the cone of u that contains w and let C_j^w be the cone of w that contains u. When we look at two vertices u and w in the constrained Y_k-graph, we notice that when the angle between uw and the bisector of C_i^u is α, the angle between wu and the bisector of C_j^w is $\theta/2 - \alpha$ (see Figure 6). Hence, when bounding the worst case spanning ratio of constrained Y_k-graphs with an odd number of cones, we can assume without loss of generality that the angle between the bisector of the cone and uw is at most $\theta/4$.

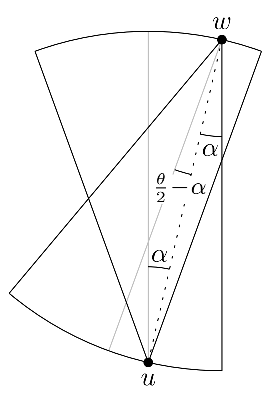

Figure 6: The angle between uw and the bisector of C_i^u is α and the angle between wu and the bisector of C_j^w is $\theta/2 - \alpha$

Theorem 3 *For odd values of $k \geq 7$, the constrained Y_k-graph is a $1/\left(1 - 2\sin\left(\frac{3\theta}{8}\right)\right)$-spanner of $Vis(P, S)$.*

Proof. Let u and w be two vertices that can see each other. We show that there exists a path connecting u and w in the constrained Y_k-graph ($k \geq 7$) of length at most $t \cdot |uw|$ for $t = 1/(1 - 2\sin(3\theta/8))$, by induction on the distance between every pair of vertices u and w that can see each other. For ease of exposition, we assume without loss of generality that $w \in C_0^u$. We also assume without loss of generality that the angle between the bisector of C_0^u and uw is at most $\theta/4$.

Base case: Vertices u and w are a closest visible pair. Using the same argument as in Theorem 2, it follows that uw is an edge of the constrained Y_k-graph and thus there exists a path between u and w of length $|uw| < t \cdot |uw|$.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices that can see each other and whose distance is less than $|uw|$.

If uw is an edge in the constrained Y_k-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between u and w, let v be the closest visible vertex to u in the subcone of u that contains w, and let x be the point along uw such that $|uv| = |ux|$ (see Figure 7). Since x lies on uw, both ux and xw are visibility edges.

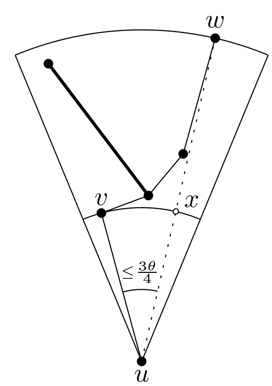

Figure 7: A convex chain from v to w

Using the same argument as in Theorem 2, it follows that vx is also a visibility edge. Hence, we can apply Lemma 1 to triangle vxw and we obtain a convex chain of visibility edges $v = p_0, ..., p_j = w$ connecting v and w (see Figure 7). Since we have at least 7 cones, the distance between any consecutive pair of vertices is strictly less than $|uw|$. Hence, since every consecutive pair of vertices along this convex chain can see each other, we can apply induction on each of them. Therefore, there exists a path from u to w via v of length at most

$$|uv| + t \cdot \sum_{i=0}^{j-1} |p_i p_{i+1}|.$$

Analogous to Theorem 2, this expression can be upper bounded by $|uv| + t \cdot (|vx| + |xw|)$.

Since $|uv| = |ux|$, triangle uvx is an isosceles triangle and we can express $|vx|$ as $2\sin(\angle vux/2) \cdot |uv|$. Since this function is increasing for $\angle vux \in [0, 2\pi/7]$ and $\angle vux$

is at most $3\theta/4$, it follows that $|vx| \leq 2\sin(3\theta/8) \cdot |uv|$. Analogous to Theorem 2, it holds that $|xw| = |uw| - |uv|$. Hence, the path between u and w has length at most

$$\begin{aligned} & |uv| + t \cdot (|vx| + |xw|) \\ \leq\ & |uv| + t \cdot \left(2\sin\left(\frac{3\theta}{8}\right) \cdot |uv| + |uw| - |uv|\right) \\ =\ & t \cdot |uw| + \left(1 + 2\sin\left(\frac{3\theta}{8}\right) \cdot t - t\right) \cdot |uv|. \end{aligned}$$

Hence, for the length of the path to be at most $t \cdot |uw|$, we need that

$$1 + 2\sin\left(\frac{3\theta}{8}\right) \cdot t - t \leq 0,$$

which can be rewritten to

$$t \geq \frac{1}{1 - 2\sin\left(\frac{3\theta}{8}\right)},$$

completing the proof. □

4 Conclusion

We showed that constrained Yao-graphs with at least 7 cones are spanners of the visibility graph and the upper bounds on the spanning ratio we obtained match those of the unconstrained Yao-graphs. This raises a number of new questions, the obvious one being whether we can reduce the upper bounds or find matching lower bound constructions.

Another set of open problems involves constrained Yao-graphs with at most 6 cones. In the unconstrained setting, it is known that the Y_k-graph is a spanner if and only if $k \geq 4$. Since the proof presented in this paper can be applied only to Yao-graphs with at least 7 cones, it remains unknown whether this is also true in the constrained setting.

Finally, though we have upper bounds on the spanning ratio of constrained Yao-graphs with at least 7 cones, we do not have a local competitive routing algorithm to actually route messages between any two visible vertices. The main difficulty stems from the inductive steps along the convex chain, since these steps make it unclear where the routing algorithm should forward the message to. In particular, we cannot assume that there exists an edge in the subcone that contains the destination, since visibility may be blocked by a constraint. Hence, routing remains a major open problem in this area.

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