Covering Grids by Trees

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Abstract

Given n points in the plane, a *covering tree* is a tree whose edges are line segments that jointly cover all the points. Let G_n^d be a $n \times \cdots \times n$ grid in $\mathbb{Z}^d$. It is known that G_n^3 can be covered by an axis-aligned polygonal path with $\frac{3}{2}n^2 + O(n)$ edges, thus in particular by a polygonal tree with that many edges. Here we show that every covering tree for the n^3 points of G_n^3 has at least $(1+c_3)n^2$ *edges*, for some constant $c_3 > 0$. On the other hand, there exists a covering tree for the n^3 points of G_n^3 consisting of only n^2+n+1 *line segments*, where each segment is either a single edge or a sequence of collinear edges. Extensions of these problems to higher dimensional grids (i.e., G_n^d for $d \geq 3$) are also examined.

1 Introduction

Let S be a set of n points in $\mathbb{R}^d$. A *covering tree* for S is a tree T drawn in $\mathbb{R}^d$ with straight-line edges such that every point in S is a vertex of T or lies on an edge of T. Similarly, a *covering path* for S is a polygonal path P drawn in $\mathbb{R}^d$ with straight-line edges such that every point in S is a vertex of P or lies on an edge of P. In this paper we study covering trees and paths for grids in $\mathbb{R}^d$.

Let the *grid* $G_{n_1,\ldots,n_d}$ denote the set of points with integer coordinates (i.e., grid points) in the hypercube $[1,n_1] \times \cdots \times [1,n_d]$ in $\mathbb{R}^d$. For simplicity we write G_n^d for the *symmetric* grid $G_{n,\ldots,n} \subset \mathbb{Z}^d$. In this paper we restrict ourselves to symmetric grids.

For the square grid G_n^2 in the plane, Kranakis et al. [6] showed that every axis-aligned covering path has at least $2n-1$ edges (a.k.a. links), and this bound can be attained. If one allows edges of arbitrary orientation in the path, Collins [4] showed that the number of links can be reduced by one: every covering path for the n^2 points of G_n^2 has at least $2n-2$ edges, and again, this bound can be attained. Recently Keszegh [5] has extended this result to covering trees: every covering tree for the n^2 points of G_n^2 has at least $2n-2$ edges; again, this bound can be attained.

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Further, it is known [3, 6] that G_n^3 can be covered by an axis-aligned polygonal path with $\frac{3}{2}n^2 + O(n)$ edges, thus in particular by a polygonal tree with that many edges. For paths, this bound is tight up to the lower order term [1]. Moving to higher dimensions, it is known [1] that G_n^d can be covered by an axis-aligned polygonal path with $(1+\frac{1}{d-1})n^{d-1} + O(n^{d-3/2})$ edges, thus in particular by a polygonal tree with that many edges; on the other hand, any axis-aligned polygonal path must consist of at least $(1+\frac{1}{d})n^{d-1} - O(n^{d-2})$ edges.

The problem of estimating the number of links needed in a covering path for the grid G_n^d appears in the collection of research problems by Braß, Moser, and Pach [2, Ch. 10.2] and in the survey article by Maheshwari et al. [7].

In this paper we investigate whether better bounds can be obtained if one allows edges of arbitrary directions in the respective covering paths or trees; no such results are known. We start with dimension 3, i.e., $d = 3$. Since every line segment (moreover, every line) covers at most n points of G_n^3, it trivially follows that every covering tree for the n^3 points of G_n^3 has at least $n^3/n = n^2$ edges. Here we show that this ideal situation is not realizable, that is, every covering tree for G_n^3 requires $\Omega(n^2)$ additional edges beyond the trivial lower bound of n^2. In particular, every covering path for the n^3 points of G_n^3 requires $\Omega(n^2)$ additional edges beyond the trivial lower bound of n^2. This gives partial answers to two questions raised by Keszegh [5].

Theorem 1 *Let $n \geq 10^3$. Every covering tree for the n^3 points of G_n^3 has at least $1.0025\, n^2$ edges. In particular, every covering path for the n^3 points of G_n^3 has at least $1.0025\, n^2$ edges.*

Our bound is quite far from the current upper bound of $\frac{3}{2}n^2 + O(n)$, which we suspect is closer to the truth. Slightly smaller multiplicative constant factors can be deduced for small n ($n \leq 10^3$) and slightly larger multiplicative constant factors can be deduced for larger n. The result in Theorem 1 can be extended to arbitrary fixed dimension d using similar methods; we omit the details.

Theorem 2 *Every covering tree for the n^d points of G_n^d has at least $(1+c_d)n^2$ edges, where $c_d > 0$ is a constant depending only on d.*

Minimizing the number of line segments. Instead of minimizing the number of edges in a covering tree, one can try to minimize the number of line segments, where each segment is either a single edge or a sequence of several collinear edges of the tree. Equivalently, one would like to determine the minimum number of segments in a connected arrangement[1] of segments that contains all points of G_n^d. Indeed, the segments of a covering tree form a connected arrangement; and an appropriate spanning tree of a connected arrangement of segments gives a covering tree for G_n^d.

The trivial lower bound of n^{d-1} also applies to the number of line segments in a covering tree. For $n = 2$, the trivial lower bound is tight, as the vertices of the hypercube G_2^d can be covered by 2^{d-1} diagonals that meet at the center. For $n \geq 3$, we show that every connected arrangement of segments that covers G_n^d requires $\Omega(n^{d-2})$ additional segments beyond the trivial lower bound of n^{d-1}, and this bound is the best possible apart from constant factors.

Theorem 3 *For every $d, n \in \mathbb{N}$, $n \geq 3$, every connected arrangement of line segments that contains G_n^d has at least $n^{d-1}+c'_d\, n^{d-2}$ segments, where $c'_d > 0$ is a constant depending only on d.*

For every $n, d \in \mathbb{N}$, there exist a connected arrangement of $(n^d-1)/(n-1) = n^{d-1}+n^{d-2}+\ldots+1$ segments that contain G_n^d; in particular, there exists a covering tree for G_n^d with that many segments.

Conjectures. Kranakis et al. [6] conjectured that, for all $d \geq 3$, every axis-aligned covering path for G_n^d consists of at least $\frac{d}{d-1}\, n^{d-1} - O(n^{d-2})$ edges. As discussed above, the conjecture has been confirmed [1, 4, 6] up to $d = 3$. It can be further conjectured [2, Chapter 10.2, Conjecture 5] that every (not necessarily axis-aligned) covering path for G_n^d consists of at least $\frac{d}{d-1}\, n^{d-1} - O(n^{d-2})$ edges. As discussed above, this stronger version has been only confirmed [5] up to $d = 2$.

2 Minimizing the Number of Edges: Proof of Theorem 1

Let T be a tree that covers the n^3 points of G_n^3. We can assume that T is contained in $[-z, z]^3$ for some suitable $z > 0$. Denote by $e(T)$ the number of edges in T. Clearly T consists of at least n^2 edges. Let $\alpha, \beta \in (0, 1)$ be two parameters we set with foresight to

$$\alpha = 0.020 \text{ and } \beta = 0.874. \tag{1}$$

We say that an edge e of T is *heavy* if it covers at least $(1-\alpha)n$ points, and *light* otherwise. If the number of heavy edges in T is at most βn^2, then at least $(1-\beta)n^2$ edges of T are light. So in this case we have

$$\begin{aligned} e(T) &\geq \beta n^2 + \frac{(1-\beta)n^3}{(1-\alpha)n} = \left(\beta + \frac{(1-\beta)}{(1-\alpha)}\right) n^2 \\ &= \left(1 + \frac{\alpha(1-\beta)}{(1-\alpha)}\right) n^2. \end{aligned} \tag{2}$$

Assume next that the number of heavy edges in T is at least βn^2. We distinguish several cases, depending on the numbers of various types of heavy edges present in T. Observe that heavy edges can be of three types (indeed, edges with consecutive points at distance larger than 2 don't contain enough points to qualify as being heavy):

Type 1: consecutive points are at distance 1, thus the corresponding segments are axis-aligned, so these edges have 3 possible directions.

Type 2: consecutive points are at distance $\sqrt{2}$, thus the corresponding segments are diagonal in axis-orthogonal planes xoy, xoz, and yoz. Such edges have 6 possible directions, two in each of the 3 axis-orthogonal planes.

Type 3: consecutive points are at distance $\sqrt{3}$, thus the corresponding segments are diagonals in 3-space. Such edges have 4 possible directions.

Observation 1 *Let e be a non-vertical edge of T with an endpoint in the rectangular box $B = [a, n-a+1] \times [a, n-a+1] \times [-z, z]$. Then e covers at most $n - a$ points.*

Let $\beta = \beta_1 + \beta_2 + \beta_3$, where

$$\beta_1 = \beta - 12.45\alpha - 6.45\alpha^2, \quad \beta_2 = 12.45\alpha, \quad \beta_3 = 6.45\alpha^2.$$

Overall, heavy edges can have 13 possible directions. We distinguish 3 possible cases and at least one of them must occur:

Case 1. There are at least $\beta_1 n^2$ heavy edges of type 1.

Case 2. There are at least $\beta_2 n^2$ heavy edges of type 2.

Case 3. There are at least $\beta_3 n^2$ heavy edges of type 3.

We proceed with the case analysis:

Case 1: There are at least $\beta_1\, n^2$ heavy edges of type 1. Since edges of type 1 have 3 possible directions, there are at least $\beta_1 n^2/3$ heavy edges with the same direction. For convenience assume that these edges are vertical. Obviously, the vertical lines supporting these edges are all distinct.

Put[2] $a = \lfloor \alpha n \rfloor$. Observe that the number of vertical grid lines through points of $G_n^3 \setminus [a, n-a+1] \times [a, n-a+1] \times [1, n]$ is at most $4a(n-a) \leq 4an$. The supporting vertical lines of at most $4an = 4\alpha n^2$ heavy edges intersect the border of width a of $[1, n] \times [1, n]$; see

[1] An arrangement of line segments is said to be *connected* if the union of the segments is an arc-connected set.

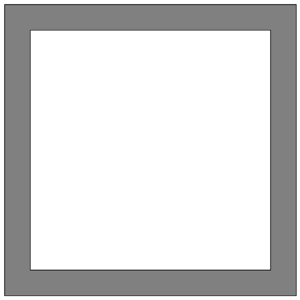

Figure 1: The border of width $a = \lfloor \alpha n \rfloor$ in G_n^3 (drawn shaded) in view from the top; the figure is not to scale.

Fig. 1. It follows that the remaining $(\beta_1/3 - 4\alpha)n^2$ vertical heavy edges are on vertical lines in the rectangular box $B = [a, n-a+1] \times [a, n-a+1] \times [-z, z]$.

Consider a standard top-down representation of the tree T with the root at the top. Color each vertical heavy edge lying in the box B blue. Since blue edges are parallel, no two share a common tree vertex. Since T is connected, by Observation 1, each blue edge is adjacent to a light edge of T. Uniquely charge each blue edge in T to the unique light edge adjacent to it on the upward path to the root of T. This charging can be applied to all blue edges except possibly to one blue edge incident to the root, if any. Since the number of blue edges is quadratic in n, this possible exception can be ignored in the counting.

It follows that at least $(\beta_1/3 - 4\alpha)n^2$ edges of T are light covering at most $(1-\alpha)n$ points. The worst case is when equality occurs, i.e., $(\beta_1/3-4\alpha)n^2$ edges can cover at most $(1-\alpha)n$ points each. The remaining points can be covered at the rate of at most n per edge. It follows that

$$\begin{aligned} e(T) &\geq \left[1 - \left(\frac{\beta_1}{3} - 4\alpha\right)(1-\alpha) + \left(\frac{\beta_1}{3} - 4\alpha\right)\right] n^2 \\ &= \left[1 + \alpha\left(\frac{\beta_1}{3} - 4\alpha\right)\right] n^2 \\ &= \left[1 + \alpha\left(\frac{\beta}{3} - 8.15\alpha - 2.15\alpha^2\right)\right] n^2. \qquad (3) \end{aligned}$$

Case 2: There are at least $\beta_2\, n^2$ heavy edges of type 2. We show that this case cannot occur. Recall that edges of type 2 have 6 possible directions along diagonals of axis-orthogonal planes. For a fixed direction in a fixed axis-orthogonal plane, the number of heavy edges parallel to the main diagonal of that plane is at most $2a+1$. Over all relevant directions and planes there are at most

$$6 \cdot (2a+1)n \leq 12an + 6n = 12\alpha n^2 + 6n$$

such edges. However $12\alpha n^2 + 6n < 12.45\alpha n^2 = \beta_2\, n^2$, which contradicts the assumption in Case 2, so this case cannot occur.

Case 3: There are at least $\beta_3\, n^2$ heavy edges of type 3. We show that this case also cannot occur, either. Recall that edges of type 3 have 4 possible directions along space diagonals of G_n^3. For a fixed diagonal direction, the number of edges parallel to this direction and covering at least $n-a$ points is at most $3\sum_{i=1}^{a} i + 1 = \frac{3a(a+1)}{2} + 1$. Over all 4 directions there are at most

$$4\,\frac{3a(a+1)}{2} + 4 = 6a(a+1) + 4$$

such edges. However,

$$\begin{aligned} 6a(a+1)+4 &= 6\alpha n(\alpha n + 1) + 4 = 6\alpha^2 n^2 + 6\alpha n + 4 \\ &< 6.45\alpha^2 n^2 = \beta_3\, n^2, \end{aligned}$$

which contradicts the assumption in Case 3, so this case also cannot occur.

To conclude the case analysis, observe that with our choice of parameters in (1), we have

$$\frac{\alpha(1-\beta)}{(1-\alpha)} \geq 0.0025, \text{ and}$$

$$\alpha\left(\frac{\beta}{3} - 8.15\alpha - 2.15\alpha^2\right) \geq 0.0025.$$

Taking into account (2) and (3), it follows that $e(T) \geq 1.0025\, n^2$, as required. This completes the proof of Theorem 1. □

3 Minimizing the Number of Segments: Proof of Theorem 3

A general upper bound. For every $d, n \in \mathbb{N}$, $n \geq 2$, we construct a covering tree $T(n,d)$ for G_n^d with

$$\frac{n^d - 1}{n-1} = n^{d-1} + n^{d-2} + \ldots + 1$$

segments. We proceed by induction on d. Refer to Fig. 2, middle. For $d = 1$, the n points of G_n^1 are collinear, and can be covered by a tree (path) with one line segment, denoted $T(n,1)$. For $d \geq 2$, note that G_n^d is the union of n translated copies of G_n^{d-1}, lying in the hyperplanes $x_d = 1, 2, \ldots, n$. Consider the covering tree $T(n, d-1)$ for the copy of G_n^{d-1} in the hyperplane $x_d = 1$. Extend this tree to a covering tree $T(n,d)$ for G_n^d by adding a segment parallel to the x_d-axis to each point of G_n^{d-1}. The number of segments in $T(n,d)$ is $n^{d-1} + (n^{d-2} + \ldots + 1)$, as claimed.

[2]For simplicity, floors and ceilings are omitted in the calculation; the resulting bounds are unaffected.

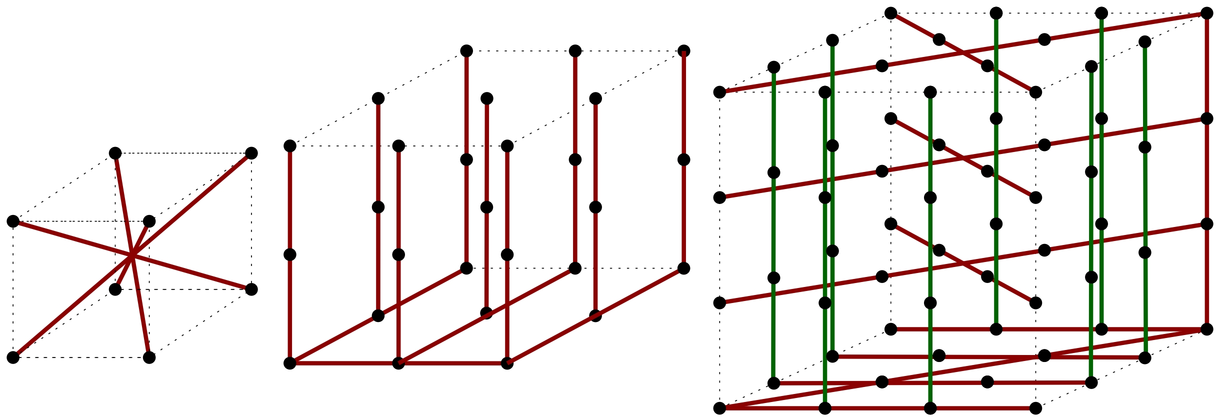

Figure 2: Covering trees for G_2^3, G_3^3, and G_4^3 in $\mathbb{R}^3$.

The special case of hypercubes. For $n = 2$, G_2^d is the vertex set of a d-dimensional hypercube; see Fig. 2, left. The 2^{d-1} space diagonals of the hypercube cover all 2^d vertices. These diagonals meet at the center of the hypercube, and so they form a covering tree (namely, a star). Since no three points in G_2^d are collinear, every covering tree consists of at least $2^d/2 = 2^{d-1}$ segments. Hence the covering tree above is optimal with respect to the number of segments.

Improved construction for n even. For $n \geq 3$, the grid G_n^d no longer admits a covering tree with only n^{d-1} segments. However, the special case of the hypercubes suggests an improved construction for n even. The key observation is that when n is even, the 2^{d-1} space diagonals of G_n^d meet in a single point, and the intersection point is not in G_n^d.

For every $d, n \in \mathbb{N}$, where $d \geq 3$ and n even, we construct a covering tree $T'(n,d)$ for G_n^d with

$$\frac{n^d - 1}{n - 1} - 2^{d-1} + d$$

segments. We proceed by induction on d. For $d = 3$, the grid G_n^3 consists of n disjoint copies of G_n^2 in n horizontal planes $z = 1, 2, \ldots, n$. Cover the n^2 points of G_n^2 in the plane $z = 1$ by a tree with $n + 1$ segments that consists of n parallel segments and a diagonal of G_n^2. (Refer to Fig. 2, right.) In each of the other $n - 1$ horizontal copies of G_n^2, the two main diagonals cover $2n$ points, and a single vertical segment connects these pairs of diagonals to a point in the plane $z = 1$ (using $2(n-1) + 1 = 2n - 1$ segments). We still need to cover $n^2 - 2n$ off-diagonal points in each of $n - 1$ horizontal copies of G_n^2. We cover these points by $n^2 - 2n$ vertical line segments, each of which is attached to the tree in the plane $z = 1$. We obtain a covering tree $T'(n, 3)$ with

$$(n + 1) + (2n - 1) + (n^2 - 2n) = n^2 + n$$

segments.

For $d \geq 4$, the grid G_n^d consists of n disjoint copies of G_n^{d-1} that lie in parallel hyperplanes $x_d = 1, 2, \ldots, n$. Cover the copy of G_n^2 in the hyperplane $x_d = 1$ by a covering tree $T'(n, d-1)$. In each of the other $n - 1$ parallel copies of G_n^{d-1}, the 2^{d-2} main diagonals cover $2^{d-2}n$ points. A single segment parallel to the x_d-axis connects the stars formed by these diagonals to an arbitrary point in the hyperplane $x_d = 1$. The remaining $n^{d-1} - 2^{d-2}n$ off-diagonal points in each of these $n - 1$ copies of G_n^2 are covered by $n^{d-1} - 2^{d-2}n$ segments parallel to the x_d-axis. The total number of segments in the resulting covering tree $T'(n, d)$ is

$$\begin{aligned}
& \left((n^{d-2} + \ldots + 1) - 2^{d-2} + (d-1)\right) \\
& +(2^{d-2}(n-1) + 1) + (n^{d-1} - 2^{d-2}n) \\
= \; & (n^{d-1} + \ldots + 1) - 2^{d-2} + (d-1) - 2^{d-2} + 1 \\
= \; & (n^{d-1} + \ldots + 1) - 2^{d-1} + d,
\end{aligned}$$

as claimed. This completes the induction step for the construction of $T'(n, d)$. Note that for $d = 3$, the two expressions match, i.e.,

$$n^2 + n = (n^{d-1} + \ldots + 1) - 2^{d-1} + d.$$

Lower bound. Let $\mathcal{L}$ be a connected arrangement of line segments in $\mathbb{R}^d$ that contains all points of the grid G_n^d. The following greedy procedure orders the segments in $\mathcal{L}$ from 1 to $m = |\mathcal{L}|$. Let ℓ_1 be an arbitrary segment in $\mathcal{L}$ that contains the maximum number of points of G_n^d, say $n_1 = \ell_1 \cap G_n^d$. For $i = 2, \ldots, m$, let ℓ_i be a segment in $\mathcal{L} \setminus \{\ell_1, \ldots, \ell_{i-1}\}$ that meets one of the previous segments $\ell_1, \ldots, \ell_{i-1}$ and contains the maximum number, say n_i, of uncovered points, that is, points in $G_n^d \setminus (\ell_1 \cup \ldots \cup \ell_{i-1})$. By construction, we have $n^d = \sum_{i=1}^{m} n_i$.

A covering tree T has two types of vertices: (1) vertices that lie at points of G_n^d and (2) *Steiner points* that are not in G_n^d. The following proposition indicates that Steiner points play a crucial role in minimizing the number of segments in a covering tree.

Proposition 4 *If every vertex of a covering tree T for G_n^d is a point in G_n^d, then T has at least $\frac{n^d-1}{n-1} = n^{d-1} + n^{d-2} + \ldots + 1$ segments. This bound is the best possible.*

Proof. Let $\mathcal{L}$ be the set of line segments in T, ordered by the greedy procedure above. Every line segment contains at most n points of G_n^d, hence $n_i \leq n$ for all i. For $i \geq 2$, however, the intersection point of ℓ_i with a previous segment is a point in G_n^d, which is already covered by some previous segment. Consequently, $n_i \leq n-1$ for $i = 2, \ldots, m$. That is, $n^d = \sum_{i=1}^m n_i \leq m(n-1)+1$, and so $m \geq (n^d-1)/(n-1)$, as required.

The tightness of the bound follows from the general upper bound given in the first paragraph of this section. □

To derive a lower bound on the number of segments in an arbitrary covering tree, we introduce some terminology in relation to G_n^d. We say that a line in $\mathbb{R}^d$ is *heavy* if it contains more than $\lceil n/2 \rceil$ points of G_n^d, and it is *full* if it contains n points of G_n^d. Let $B = \prod_{i=1}^d [1, n]$ denote the bounding box of G_n^d. We need a few easy observations.

Observation 2

1. *Every full line for G_n^d contains a diagonal of a copy of G_n^k within G_n^d, for some $k = 1, 2, \ldots, n$ (the diagonals of a copy of G_n^1 are axis-parallel).*
2. *Every full line for G_n^d is either axis-parallel or contained in one of $2\binom{d}{2}$ hyperplanes of the form $x_i - x_j = 0$ or $x_i + x_j = n+1$, where $1 \leq i < j \leq d$.*
3. *Every heavy line is parallel to a full line, and every axis-parallel heavy line is full.*

For the charging scheme in the proof of Theorem 3, we need to control the number of full lines that intersect a single line segment.

Proposition 5 *Let $d \in \mathbb{N}$ be a constant.*

1. *Every line in $\mathbb{R}^d$ intersects $O(n)$ full lines for G_n^d.*
2. *Every heavy line for G_n^d contains at most one Steiner point that is not in $\mathbb{Z}^d$ but lies on some other full line for G_n^d.*
3. *A heavy line containing $n-a$ points of G_n^d intersects $O(a)$ full lines at Steiner points in $\mathbb{Z}^d \setminus G_n^d$.*
4. *Every Steiner point lies on at most 2^{d-1} full lines.*

Proof. (1) By Observation 2, the full lines have

$$\sum_{k=1}^{d} \binom{d}{k} 2^{k-1} = \frac{3^d - 1}{2}$$

different orientations, and so they can be partitioned into $(3^d-1)/2$ families of parallel lines. Every line ℓ in $\mathbb{R}^d$ meets at most n full lines from each parallel family, thus $O(n)$ full lines overall.

(2) Assume that a heavy line ℓ intersects a full line ℓ' and the intersection point $\ell \cap \ell'$ is not in the integer lattice $\mathbb{Z}^d$. The point $\ell \cap \ell'$ lies in some unit cube σ spanned by $\mathbb{Z}^n$ (where $\ell \cap \ell'$ is either in the interior or on the boundary of σ). By assumption, $\ell \cap \ell'$ is not a vertex of σ, hence both $\ell \cap \sigma$ and $\ell' \cap \sigma$ are line segments. By Observation 2(3), the heavy line ℓ is parallel to a full line, which is the diagonal of a copy of G_n^k in G_n^d for some $1 \leq k \leq d$. Therefore, ℓ contains a diagonal of a k-dimensional face of σ, and similarly ℓ' contains the diagonal of a k'-dimensional face of σ for some $1 \leq k' \leq d$. However, the diagonals of any two different faces of the unit cube σ are either disjoint or meet at a vertex of σ. Consequently, both ℓ and ℓ' contain diagonals of the same k-dimensional face of σ. The full line ℓ' must be a diagonal of the copy of $G_n^k \subseteq G_n^d$ that spans this particular k-face of σ. That is, both ℓ and ℓ' lie in the same copy of $G_n^k \subseteq G_n^d$, and ℓ has a unique intersection point $\ell \cap \ell'$ with the diagonals of this copy of G_n^k.

(3) Let ℓ be a heavy line passing through $n-a$ points of G_n^d, and let ℓ' be a full line such that $\ell \cap \ell'$ is in $\mathbb{Z}^d \setminus G_n^d$. Then every full line parallel to ℓ' is either disjoint from ℓ or intersects ℓ in an integer point in $\mathbb{Z}^d$. The lines ℓ and ℓ' span a 2-dimensional plane P. The plane P contains n full lines parallel to ℓ', but $n-a$ of them meet ℓ' at points in G_n^d. Consequently, at most a of these lines intersect ℓ outside of G_n^d. Summing over all $(3^d-1)/2 = O(1)$ directions of full lines, at most $O(a)$ full lines intersect ℓ at points in $\mathbb{Z}^d \setminus G_n^d$.

(4) If two full lines meet in a Steiner point p, then p is the center of a copy of $G_n^k \subset G_n^d$, for some $2 \leq k \leq d$. The only full lines incident to p are the $2^{k-1} \leq 2^{d-1}$ diagonals of this copy of G_n^k. □

Proof of Theorem 3. Let $\mathcal{L}$ be a connected arrangement of line segments that cover G_n^d. Order the segments as $(\ell_1, \ldots, \ell_m)$ by the greedy procedure described earlier. (That is, ℓ_1 is a segment in $\mathcal{L}$ that contains the maximum number n_1 of points of G_n^d; and ℓ_i, for $i = 2, \ldots, m$, meets one of the previous segments and contains the maximum number n_i of uncovered points of G_n^d.) We show that the average of n_i, the number of "new" points covered by segment ℓ_i, is $n - \Omega(1)$. We distinguish three types of segments in $\mathcal{L}$.

- $\mathcal{L}_1 = \{\ell_i \in \mathcal{L} : n_i = n\}$;
- $\mathcal{L}_2 = \{\ell_i \in \mathcal{L} : \lceil n/2 \rceil < n_i < n\}$;
- $\mathcal{L}_3 = \{\ell_i \in \mathcal{L} : n_i \leq \lceil n/2 \rceil\}$.

Partition the sequence $(\ell_1, \ldots, \ell_m)$ into maximal subsequences of consecutive elements such that each segment in $\mathcal{L}_2 \cup \mathcal{L}_3$ is the first element of a subsequence.

Consider one such subsequence $(\ell_i, \ldots, \ell_{i+k})$, where $k \geq 0$. By construction, $\ell_i \in \mathcal{L}_2 \cup \mathcal{L}_3 \cup \{\ell_1\}$, and $\ell_{i+1}, \ldots, \ell_{i+k} \in \mathcal{L}_1$. The full lines $\ell_{i+1}, \ldots, \ell_{i+k}$ each intersect a previous segment at a Steiner point. Due to the greedy ordering, they each intersect a previous segment from the same subsequence. By Proposition 5(2), a full line meets any other full lines at the same Steiner point. Consequently, every line $\ell_{i+1}, \ldots, \ell_{i+k}$ meets ℓ_i or a previous full line of the subsequence which meets ℓ_i. Each full line that meets ℓ_i is responsible for at most 2^{d-1} other full lines (that do not meet ℓ_i) by Proposition 5(4). Therefore, at least $k/2^{d-1} = \Omega(k)$ full lines in $\ell_{i+1}, \ldots, \ell_{i+k}$ meet ℓ_i in Steiner points.

If $\ell_i \in \mathcal{L}_2$, then ℓ_i is contained in a heavy line, which contains $n - a$ points for some $0 \leq a < \lfloor n/2 \rfloor$. That is, $n_i \leq n - \max(1, a)$. By Proposition 5(3), ℓ_i meets at most $O(a + 1)$ full lines in Steiner points (inside or outside of B). This implies $k = O(a + 1)$, and so the segments $\ell_i, \ldots, \ell_{i+k}$ contain an average of at most

$$\frac{kn + n - \max(1, a)}{k + 1} = n - \frac{\max(1, a)}{k + 1} = n - \Omega(1)$$

new points.

If $\ell_i \in \mathcal{L}_3$, then ℓ_i contains at most $n/2$ points of G_n^d and it meets $O(n)$ full lines by Proposition 5(1). In this case, the segments $\ell_i, \ldots, \ell_{i+k}$ contain an average of at most

$$\frac{kn + n/2}{k + 1} = n - \frac{n}{2(k + 1)} = n - \Omega(1)$$

new points.

Finally, if $\ell_1 \in \mathcal{L}_1$, then the average of n_i is n in the very first subsequence. In this case, by Proposition 5(4), the full line ℓ_1 meets at most $2^{d-1} - 1$ other full lines in a Steiner point, so this special subsequence covers at most $2^{d-1}n$ points of G_n^d.

Consequently, the average of n_i over all segments $\ell_i \in \mathcal{L}$ is $n - \Omega(1)$ if $n \geq 3$. Now $n^d = \sum_{i=1}^{m} n_i = m(n - \Omega(1))$ yields $m = n^{d-1} + \Omega(n^{d-2})$, as claimed. □

4 Conclusion

We conclude with a few open problems:

1. Does every covering path for G_n^3 require at least $\left(\frac{3}{2} - o(1)\right) n^2$ edges?
2. Does every covering tree for G_n^3 require at least $\left(\frac{3}{2} - o(1)\right) n^2$ edges?
3. Does every covering tree for G_n^3 require at least $n^2 + n$ segments?
4. Does every covering tree for G_n^d require at least $\dfrac{n^d - 1}{n - 1} - 2^{O(d)}$ segments?

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