Geometric Hitting Set and Set Cover Problems with Half-Strips

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Abstract

We show that hitting set and set cover problems with half-strips oriented in two opposite directions are NP-complete.

1 Introduction

A half-strip oriented in the upward direction consists of all points in the region bounded by two vertical lines which are also bounded from below by a horizontal line (see Figure 1(a)).

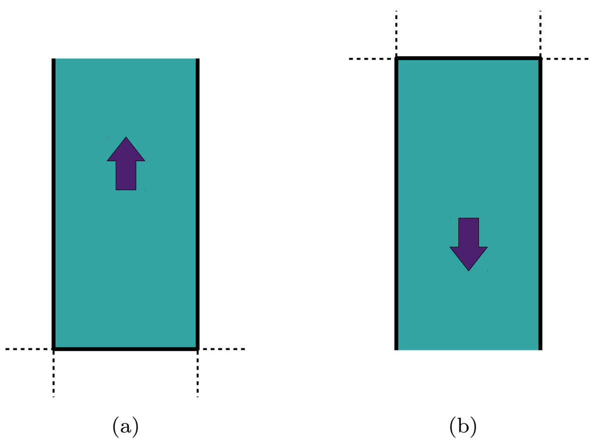

Figure 1: Half-strips oriented in (a) upward and (b) downward directions.

In this paper, we consider hitting set and set cover problems with points in the plane and half-strips oriented in upward and downward directions.

Problem 1 Hitting half-strips in two opposite directions by points (*HHS-OD*). *We are given a set P of points, a set H of half-strips oriented in two opposite directions, and a positive integer α. The goal is to decide whether there exists a subset $P' \subseteq P$ of points with size at most α that hits all the half-strips.*

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Problem 2 Covering points by half-strips in two opposite directions (*CHS-OD*). *We are given a set P of points, a set H of half-strips oriented in two opposite directions, and a positive integer β. The goal is to decide whether there exists a subset $H' \subseteq H$ of half-strips with size at most β that covers all the points.*

It is clear that both these problems are in NP. The main result of this paper is that *HHS-OD* and *CHS-OD* are NP-complete.

Previous Work. Katz et al. [5] give a polynomial time algorithm for the hitting set problem when the half-strips are oriented in one direction. Later on, Chan and Grant [3] also give a polynomial time algorithm for the same problem. For the set cover problem with half-strips in one direction, a polynomial time algorithm was given by Katz et al. [5], Chin et al. [4], Chakrabarty et al.[2], and Chan and Grant [3].

One can observe that half-strips oriented in two opposite directions are pseudodisks. Therefore, from the result of Mustafa and Ray [10] there is a PTAS for *HHS-OD* and from the result of Mustafa et al. [9] there is a QPTAS for *CHS-OD*.

Bereg et al. [1] give a factor 2 approximation for the class cover problem with half-strips in two opposite directions. A generalized version of the class cover problem for strips and half-strips was studied in [8].

2 Hitting half-strips in two opposite directions by points (*HHS-OD*)

In this section, we prove that *HHS-OD* is NP-complete by a reduction from Planar 3-SAT. Lichtenstein [7] proposed Planar 3-SAT as follows:

Definition 1 Planar 3-SAT [7, 6] *Let ϕ be a SAT formula with n variables and m clauses such that each clause contains at most 3 variables. Further, ϕ can be embedded on the plane as follows. All the variables are aligned in a horizontal line and each 3 legged (for 3 variables) and each 2 legged (for 2 variables) clause connects to the variables either from below or from above so that no two clauses intersect (see Figure 2). Now, we have to find an assignment which satisfies ϕ.*

Let ϕ be a Planar 3-SAT formula with n variables and m clauses. Let c_b be the maximum number of legs

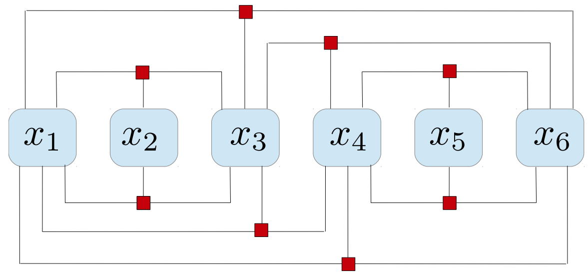

Figure 2: PLANAR 3-SAT representation.

incident on a variable by clauses from below. Also let c_t be the maximum number of legs incident on a variable by clauses from above. Now let $c = \max\{c_b, c_t\}$ and $k = c + 1$.

Variable Gadget: For each variable gadget, we take $4k$ points in two horizontal rows such that each row contains $2k$ points (see Figure 3). We connect these points by edges to form a cycle. Now we take $4k$ half-strips, each half-strip corresponding to an edge in the cycle. To hit all these half-strips, at least $2k$ points are required. Now observe that there are only two possible optimal solutions, S_1 and S_2 (see Figure 3), of size $2k$, which represent the truth values of the corresponding variable:

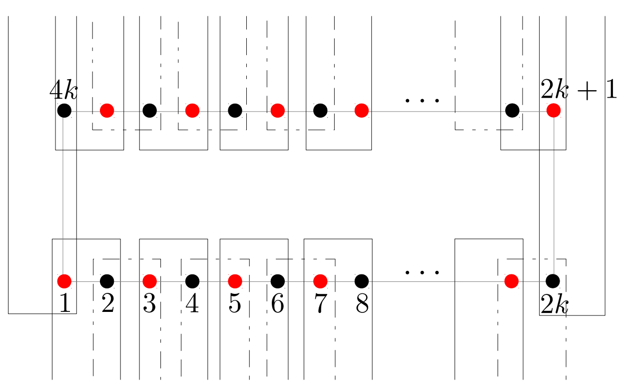

Figure 3: Variable gadget with optimal solutions S_1 (odd numbered points) and S_2 (even numbered points).

Claim 1 *There are exactly two possible optimal hitting sets, $S_1 = \{1, 3, \cdots, 4k-1\}$ and $S_2 = \{2, 4, \cdots, 4k\}$, of cost $2k$ for the variable gadget in Figure 3.*

Proof. Note that a hitting set of half-strips in the variable gadget is equivalent to a vertex cover in the cycle connecting all the points. Since the cycle contains $2k$ disjoint edges, there does not exist any solution of size at most $2k-1$. Let S be a solution of size $2k$. Consider the following two cases. Case 1: No two vertices in S are consecutive. In this case S is either S_1 or S_2. Case 2: Some vertices in S are consecutive. Without loss of generality, assume that vertices 1 and 2 are in S. Then at least $2k-1$ vertices are required to cover edges $(3,4), (5,6), \cdots, (4k-1, 4k)$. Hence, S has at least $2k+1$ vertices, which is a contradiction. □

Clause Gadget: For each clause we take one half-strip as shown in Figure 4.

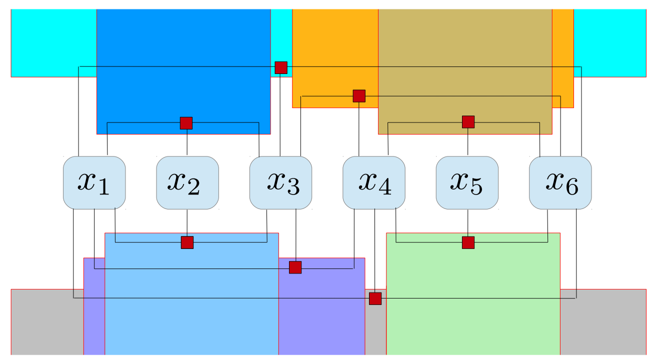

Figure 4: Clause half-strips for the formula in Figure 2.

Variable-Clause Interaction: The interaction between the variable and the clause gadgets is set up by vertically moving some points in the variable gadgets. We now describe the vertical shifting of points for the variable gadget corresponding to the variable x_i. First we fix the four end points $\{1, 2k, 2k+1, 4k\}$. Let l_i be the number of legs that connect to the variable x_i from below. Now number the clauses corresponding to these legs $C_1, C_2, \cdots, C_{l_i}$ in the order in which their legs connect to variable x_i from left to right.

We now partition the points in the bottom row of the variable gadget into pairs of consecutive points starting from the pair $\{2, 3\}$. We associate the r-th pair of points with clause C_r, where $1 \leq r \leq l_i$. If x_i occurs as a negative literal in clause C_r, vertically shift the point $2r$ to the top edge of the half-strip corresponding to clause C_r. If x_i occurs as a positive literal in C_r, vertically shift the point $2r+1$ to the top edge of the half-strip corresponding to clause C_r. A similar shifting of points is done for clauses that connect to x_i from above. In Figure 5, we demonstrate the above construction for the variable x_4 of the PLANAR 3-SAT example of Figure 2.

Thus, given a formula ϕ with n variables and m clauses, we obtain an instance (P_ϕ, H_ϕ) of *HHS-OD* with $4kn$ points and $4kn+m$ half-strips. We now assume that $\alpha = 2kn$.

Lemma 1 *ϕ is satisfiable iff there exists a solution to (P_ϕ, H_ϕ) with cost at most $2kn$.*

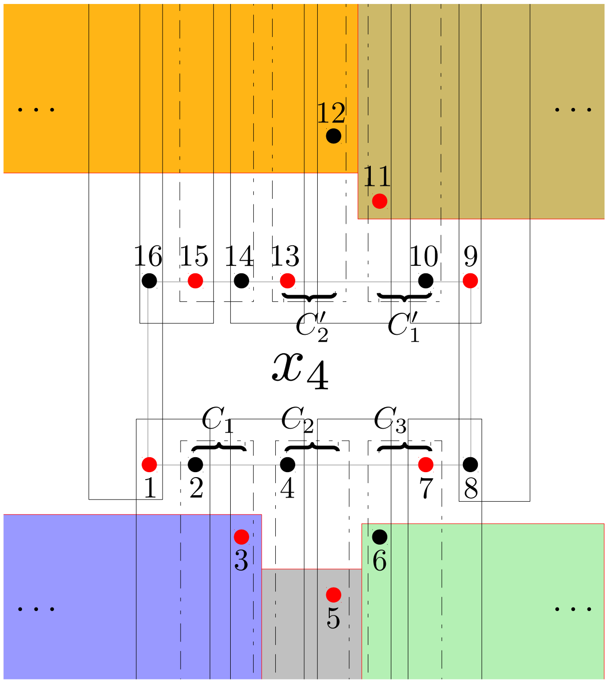

Figure 5: Vertical movement of points in the variable gadget for x_4. We assume that variable x_4 occurs as a positive literal in clauses C_1, C_2, and C_1' and as a negative literal in clauses C_3 and C_2'.

Proof. (only if part) Assume that ϕ has a satisfying assignment. For the i-th variable gadget, take the solution S_1 if variable x_i is true and S_2 if variable x_i is false. We pick a total of $2kn$ points and these points hit all the variable and clause half-strips.

(if part) Suppose there is a solution to (P_ϕ, H_ϕ) with cost at most $2kn$. To hit all the half-strips in a variable gadget requires at least $2k$ points. Note that all the variable gadgets are disjoint. Therefore, from each variable gadget we must pick exactly $2k$ points (either set S_1 or set S_2). Set variable x_i to true if S_1 is picked in the variable gadget, otherwise set x_i to false. Since each clause half-strip contains either 3 (for 3 variable clauses) or 2 (for 2 variable clauses) points shifted from variable gadgets and one of these points must be picked in any hitting set, this gives a satisfying assignment for formula ϕ. □

From Lemma 1, we have the following theorem.

Theorem 2 *HHS-OD is NP-complete.*

3 Covering points by half-strips in two opposite directions (*CHS-OD*)

In this section, we prove that *CHS-OD* is NP-complete by giving a reduction from Planar 3-SAT (see Definition 1).

Given formula ϕ, let c_b, c_t, and c be the numbers as defined in Section 2.

Variable Gadget: The variable gadget (see Figure 6) is the same as the variable gadget of *HHS-OD*, with the difference that we now take two horizontal rows of $8c+1$ points each. By an argument similar to Claim 1, we can say that there are exactly two optimal set covers of size $8c+1$: HS_1 (all odd numbered half-strips) and HS_2 (all even numbered half-strips). This gives the truth assignment of the corresponding variable.

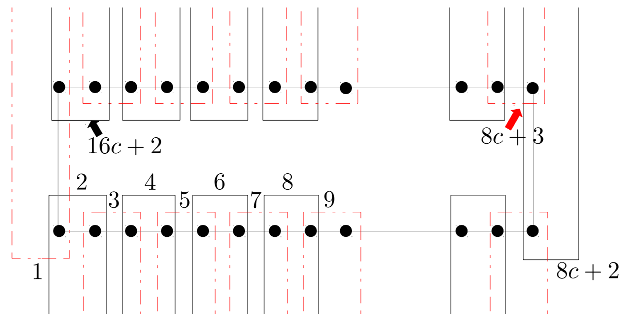

Figure 6: Variable gadget for *CHS-OD* with optimal solutions HS_1 (odd numbered half-strips) and HS_2 (even numbered half-strips).

Clause Gadget: The "regions" containing the clause gadgets are as shown in Figure 4. For a 3 variable clause with x_i, x_j, and x_k as left, middle, and right variables, the clause gadget is a set of 9 points and 4 half-strips covering these points as shown in Figure 7. Similarly, for a 2 variable clause with x_i and x_j as left and right variables, the clause gadget is a set of 5 points and 2 half-strips covering these points as shown in Figure 8.

Note that a different set of 9 or 5 points are added for each clause.

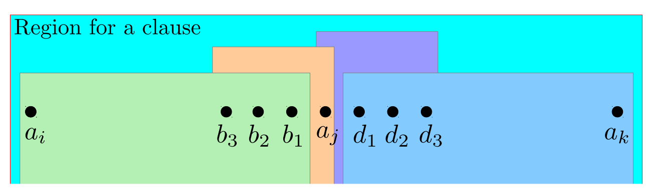

Figure 7: 3 variable clause gadget.

Variable-Clause Interaction: Each point in the clause gadgets lies in exactly one half-strip from the variable gadgets. Now we describe the alignment of points in the clause gadgets with the half-strips in the gadget for variable x_j. As before, let $C_1, C_2, \cdots, C_{l_j}$ be the clauses connecting to x_j from below. We group the downward half-strips in the variable gadget into sets of 8 consecutive half-strips starting from group $\{2, \cdots, 9\}$. We associate the r-th group with clause C_r, where $1 \leq r \leq l_j$.

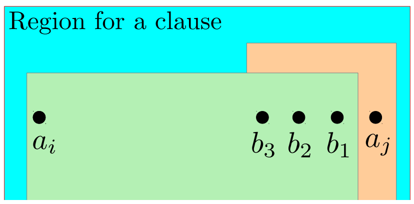

Figure 8: 2 variable clause gadget.

Now consider a 3 variable clause C_r, with x_j as a middle variable. If x_j occurs as a negative literal, place the seven middle points as shown in Figure 9. If x_j occurs as a positive literal, place the seven middle points as shown in Figure 10. Now suppose x_j is a left or right variable in C_r. Place point a_j so that it aligns with an odd numbered half-strip from the group of half-strips for clause C_r if x_j occurs as a negative literal and with an even numbered half-strip from the group of half-strips for clause C_r if x_j occurs as a positive literal.

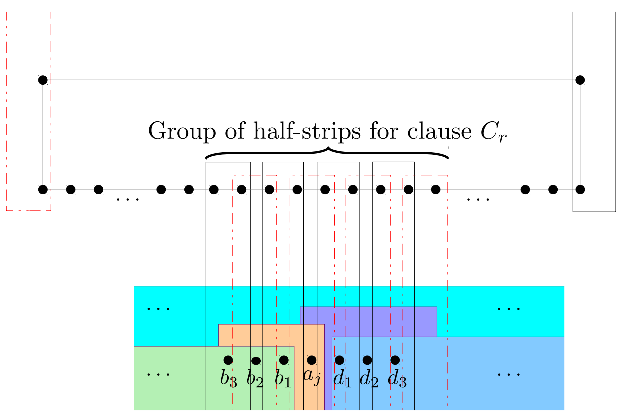

Figure 9: Placement of clause points and half-strips for negative middle variable x_j.

For a 2 variable clause C_r with x_j as a right variable, the placement of $\{b_3, b_2, b_1, a_j\}$ is the same as that in Figures 9 and 10 with $\{d_1, d_2, d_3\}$ removed. If x_j is a left variable, place a_j in an odd or even numbered half-strip from the group of half-strips for clause C_r according to whether x_j occurs as a negative or positive literal.

This completes the construction. Thus, given a formula ϕ with n variables, m_1 3 variable clauses, and m_2 2 variable clauses we construct an instance (P_ϕ, H_ϕ) of *CHS-OD* with $(16c + 2)n + 9m_1 + 5m_2$ points and $(16c + 2)n + 4m_1 + 2m_2$ half-strips. Here we assume that $\beta = (8c + 1)n + 2m_1 + m_2$. Now we prove the following lemma:

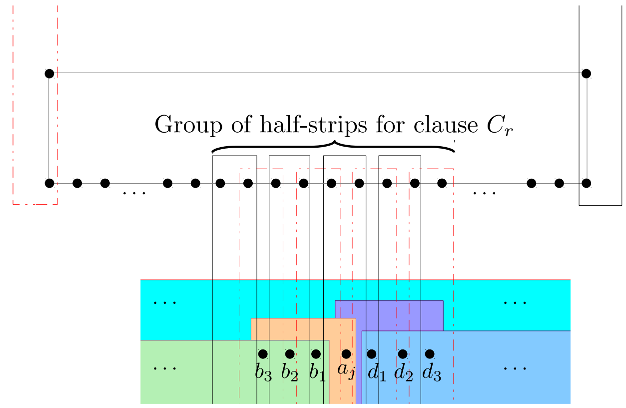

Figure 10: Placement of clause points and half-strips for positive middle variable x_j.

Lemma 3 *ϕ is satisfiable iff there exists a solution to (P_ϕ, H_ϕ) with cost at most $(8c+1)n + 2m_1 + m_2$, where m_1 and m_2 are the number of clauses that contain 3 variables and 2 variables respectively.*

Proof. (only if part) Assume that ϕ is satisfiable. In the variable gadget for x_i, take the solution HS_1 if x_i is false and HS_2 if x_i is true. Thus, we are taking $(8c+1)n$ half-strips from variable gadgets. Now for each 3 variable clause at least one of a_i, a_j, or a_k is covered. Then, 2 half-strips are sufficient to cover the remaining points from the clause gadget. For each 2 variable clause, at least one of a_i or a_j is covered and 1 half-strip is enough to cover the remaining points. Therefore, in total $(8c + 1)n + 2m_1 + m_2$ half-strips are sufficient to cover all the points in P_ϕ.

(if part) Suppose there is a solution Sol to *CHS-OD* on (P_ϕ, H_ϕ) with cost at most $(8c+1)n+2m_1+m_2$. We now modify Sol so that at least 2 half-strips are picked from a 3 variable clause gadget and at least 1 half-strip is picked from a 2 variable clause gadget.

If Sol contains 1 half-strip from a 3 variable clause gadget, exactly one of the two triplets $\{b_1, b_2, b_3\}$ or $\{d_1, d_2, d_3\}$ are covered completely by half-strips from variable gadgets. Suppose $\{b_1, b_2, b_3\}$ is a triplet of this type. Then we can remove the half-strip from the variable gadget covering the middle point b_2 and add a half-strip from the clause gadget covering the triplet $\{b_1, b_2, b_3\}$. If Sol contains no half-strips from a 3 variable clause gadget, we will do the above modification for both the triplets $\{b_1, b_2, b_3\}$ and $\{d_1, d_2, d_3\}$.

Similarly, if no half-strips are picked from a 2 variable clause gadget, we can remove the half-strip covering point b_2 and replace it by a half-strip from the clause gadget covering $\{b_1, b_2, b_3\}$.

The above process does not increase the cost of the solution and it remains feasible. The modified Sol has exactly $(8c + 1)$ half-strips from each variable gadget, exactly 2 half-strips from each 3 variable clause gadget, and exactly 1 half-strip from each 2 variable clause gadget. The satisfying assignment is obtained by setting x_i to false iff HS_1 is picked in the corresponding variable gadget. □

We thus have the following theorem.

Theorem 4 *CHS-OD is NP-complete.*

4 Acknowledgement

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