Squeeze-Free Hamiltonian Paths in Grid Graphs

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Abstract

Motivated by multi-robot construction systems, we introduce the problem of finding squeeze-free Hamiltonian paths in grid graphs. A Hamiltonian path is *squeeze-free* if it does not pass between two previously visited vertices lying on opposite sides. We determine necessary and sufficient conditions for the existence of squeeze-free Hamiltonian paths in staircase grid graphs. Our proofs are constructive and lead to linear time algorithms for determining such paths, provided that they exist.

1 Introduction

We introduce a problem motivated by collective construction systems in which a large number of simple, autonomous robots build complex structures using modular building blocks. Such systems are inspired by the decentralized construction methods of termites and bees in which global structure emerges from the efforts of individual insects following seemingly simple rules and using environmental cues. They are robust to failure because damaged robots are easily replaced, making them suitable in uncertain and inhospitable environments.

In the TERMES collective construction system introduced in [6], the modular building blocks are cubes that are placed on a regular grid to form lattice-based structures. Robots move from cell to cell on the grid while carrying a block, which they can attach to the structure at an adjacent cell. Physical limitations restrict the class of structures that the robots can build and the order in which they can attach the blocks. For example, it is impossible for a robot to carry blocks down a corridor one block wide, or to place a block directly between two others. Inappropriate intermediate configurations that can no longer be traversed by the robots should therefore be avoided by proper robot coordination.

Coordination in the TERMES system is achieved by precomputing a path that all robots follow while adding blocks to the structure. The path starts at a grid cell on the boundary of the final structure, visits each grid cell of the final structure exactly once, and satisfies the restriction that the path may not "squeeze" into a cell that has two previously visited cells adjacent to it on opposite sides. We call such a path a *squeeze-free Hamiltonian path*. The problem thus reduces to finding a squeeze-free Hamiltonian path in a grid graph. See Figure 1.

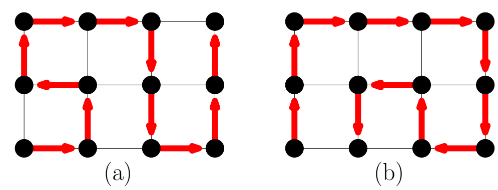

Figure 1: (a) A squeeze-free path, and (b) a path in which the last three vertices are squeezed between two previously visited vertices on opposite sides.

To our knowledge we are the first to study the algorithmic complexity of the squeeze-free Hamiltonian path problem. In [6], their main focus is on engineering the robots rather than algorithms, so they use an exponential time backtracking algorithm to compute the paths. Computing Hamiltonian cycles in general grid graphs is known to be NP-Complete [4], although the problem can be solved in polynomial time for specialized classes of grid graphs [5], [2]. See [1] for a survey and new results on Hamiltonicity of square, triangular, and hexagonal grid graphs. In other related work, [3] presents an $O(n^2)$ algorithm for computing a partial ordering on the placement of blocks subject to the squeeze-free constraint for 2D structures with holes. Here we take a first step to understanding the complexity of the squeeze-free Hamiltonian path problem by providing an $O(t)$ algorithm that determines for any *staircase* grid graph G with t steps, if G has a squeeze-free Hamiltonian path that starts at a boundary vertex located on a step of G.

2 Notation and Definitions

A grid graph is a graph induced by a finite subset of the vertices of a square tiling of the plane. In this paper we consider *staircase* grid graphs which consist of the edges and vertices bounding a set of tiles whose union forms the shape of a staircase extending rightwards and upwards from the bottom leftmost vertex.

For a staircase grid graph G, let ∂G denote the portion of the staircase boundary that extends clockwise from the bottom leftmost vertex to the top rightmost vertex of G. Refer to Figure 2. For any pair of vertices $a, b \in \partial G$, let $\partial G[a, b]$ denote the portion of ∂G extending from a to b. For any vertex x, let x_s, x_w, x_n

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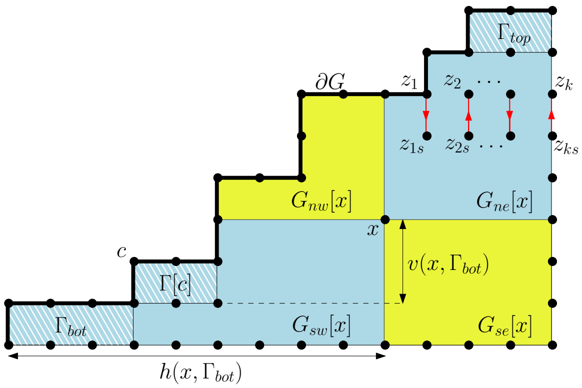


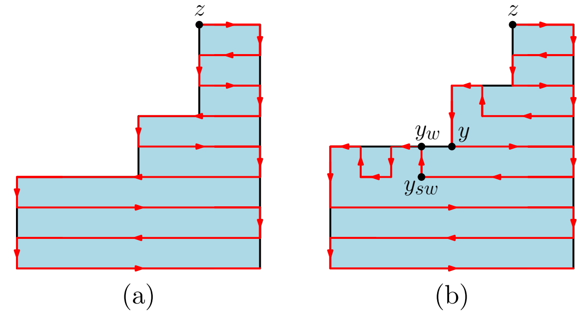
Figure 2: Staircase grid graph. Only some of the interior grid vertices and edges are illustrated.

and x_e refer to the vertices adjacent to x that lie south, west, north and east of x, respectively (if such a vertex exists). We sometimes double the subscript to refer, for instance, to the vertex south of x_e as x_{es}. At each vertex $x \in G$, the coordinate axes with origin x partition the plane into four quadrants. Let $G_{ne}[x]$, $G_{nw}[x]$, $G_{sw}[x]$ and $G_{se}[x]$ denote the subgraph of G that lies entirely in the first, second, third and fourth quadrant, respectively. We assume that each quadrant is closed, so it includes the points on the bounding axes. Define $G_n[x] = G_{ne}[x] \cup G_{nw}[x]$, and similarly for $G_s[x]$, $G_e[x]$ and $G_w[x]$. Let H be a (directed) Hamiltonian path in G. For any two vertices $a, b \in G$, such that a is visited by H before b, let $H[a, b]$ denote the directed subpath of H from a to b. A vertex $v \in H$ is *squeezed* if v_e and v_w are both visited before v, or if v_n and v_s are both visited before v.

A *corner* is a boundary vertex on G with interior angle $\pi/2$ (if convex) or $3\pi/2$ (if reflex). For any convex corner $c \in \partial G$, let $\Gamma[c]$ denote the closed rectangle bounded by the two line segments extending from c to the next and previous corners located clockwise and counterclockwise (respectively) from c. We refer to $\Gamma[c]$ as the *step* with corner c. The height of $\Gamma[c]$ is the height of the corresponding rectangle. The case where G consists of a single step is trivial, so we assume that G has at least two steps. We refer to the highest step of G as Γ_{top} and the lowest step as Γ_{bot}.

For any two vertices $a, b \in G$, let $h(a, b)$ denote the horizontal extent of the line segment ab, and let $v(a, b)$ denote the vertical extent of ab. For any vertex $x \in G$ and any staircase step Γ of G, let $h(x, \Gamma)$ denote the horizontal distance from x to the left side of Γ, and let $v(x, \Gamma)$ denote the vertical distance from x to the top side of Γ. The following definition (depicted in Figure 2 for $j = s$) will play an important role in our discussion.

Definition 1 *Let Z be a sequence of consecutive grid points $z_1, \ldots, z_k$ lying on a horizontal (vertical) grid segment. For a fixed $j \in \{s, n\}$ ($j \in \{e, w\}$), we say that the grid segments $\overline{z_1 z_k}$ and $\overline{z_{1j} z_{kj}}$ form a* zigzag sequence *in H if, for each $i = 1, \ldots, k$, $\overrightarrow{z_i z_{ij}} \in H$ if i is odd, and $\overrightarrow{z_{ij} z_i} \in H$ if i is even.*



(a) (b)

Figure 3: (a) *pattern1* and (b) *pattern2*.

We call a zigzag sequence *separating* if it extends between two boundary edges.

2.1 Hamiltonian Patterns

Let $G' \subseteq G$ be an arbitrary staircase subgraph of G. We define two distinct Hamiltonian path patterns H_1 and H_2 on G', which will later be used in stitching a Hamiltonian path H for G. Each of these patterns is defined for a fixed orientation for the first edge – say east – and grows in a particular direction – say south – with the understanding that the pattern can undergo rotations and reflections as necessary to construct H. Each of H_1 and H_2 begins at a vertex on $\partial G'$. The first pattern H_1, which we refer to as *pattern1*, includes straight horizontal path segments with orientations alternating east and west on each row, and extending between boundary points of G'. See Figure 3a. The second pattern H_2, which we refer to as *pattern2*, is identical to the first pattern, with the only difference that, for each reflex corner y, the straight horizontal subsegment extending west from y_{sw} is replaced by a subpath of unit height that includes the zigzag sequence starting with $\overrightarrow{y_{sw} y_w}$ and extending west. See Figure 3b. Observe that both patterns are squeeze-free.

3 Preliminaries

Here we prove some properties of squeeze-free Hamiltonian paths in G, if it has any. Therefore, throughout this section, assume there exists a squeeze-free Hamiltonian path in G, and H is one such path.

Lemma 1 *At any time during the traversal of H, there can be no unvisited vertices between any two vertices a and b on a grid line that have already been visited.*

Proof. If there were unvisited vertices along the line segment ab, then the vertex last visited among these vertices would cause a squeeze. $\square$

Let p be a directed, simple path in G from a boundary point x to a boundary point y. A vertex $v \in G$ is said to be *left* (*right*) of p if $v \notin p$ and v is on or to the left (right) of the oriented closed curve consisting of p followed by the path counterclockwise (clockwise) along the boundary of G from y to x.

Lemma 2 *Let x be any boundary vertex of G, and let y be a vertex of ∂G such that $y \in G_{ne}[x]$ and H visits x before y. At the time y is visited, all vertices that are left of $H[x, y]$ and also in $G_e[x]$ are visited, with the possible exception of those vertices to the left of y on its horizontal grid line.*

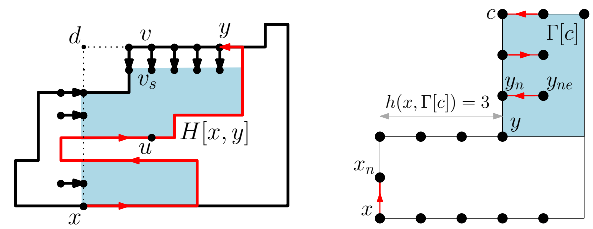
Proof. For contradiction, suppose there is an unvisited vertex z to the left of $H[x, y]$ and in $G_e[x]$ that is not left of y on y's horizontal grid line. E.g., in Figure 4a, z may be any vertex in the shaded regions. Let d be the intersection of an upward ray from x and a leftward ray from y. (Note that d need not be in G, as illustrated in Figure 4a.) To reach z from y, H must eventually either traverse an edge $\overrightarrow{vv_s}$ with v on $\overline{dy}$, or it must traverse an edge $\overrightarrow{vv_e}$ with v_e on $\overline{dx}$. (These edges, which are the only edges taking H into the shaded regions containing z, are depicted in Figure 4a.) In the first case, there must be a previously visited vertex of $H[x, y]$, call it u, located below v_s on its vertical grid line. Thus there is a time during the traversal of H in which v_s is unvisited and between visited vertices v and u in the same vertical grid line, which contradicts Lemma 1. In the second case there is a previously visited vertex of $H[x, y]$ located to the right of v_e on its horizontal grid line, which similarly contradicts Lemma 1. $\square$

Lemma 3 *The end point e of H is one of the following vertices: (i) a top corner of Γ_{top}, (ii) a left corner of Γ_{bot}, or (iii) the lowest rightmost corner of G.*

Proof. Clearly e must be a convex corner, or else it is squeezed between previously visited vertices. Suppose for contradiction that e is a convex corner of a step that is not Γ_{top} or Γ_{bot}. Then the top left vertex x of Γ_{bot} and the top left vertex y of Γ_{top} are both visited before e. Without loss of generality, assume x is visited before y. (If y is visited before x, just rotate the staircase by 90^o and reflect it across the vertical, thus reversing the roles of x and y.) By Lemma 2, e must be visited before y, a contradiction. $\square$

Lemma 4 *Let $x \in G$ be the first vertex visited by H in $G_{ne}[x]$ and let $\overrightarrow{xx_n} \in H$. Then $\overrightarrow{x_{ne}x_e} \in H$ and x_{ne} is the first vertex visited by H in $G_{se}[x_{ne}]$.*

Proof. Because x is the first vertex visited in $G_{ne}[x]$, H traverses $\overrightarrow{xx_n}$ before visiting x_e. Therefore H cannot enter x_e from the east (because x_e would be squeezed



(a) (b)

Figure 4: (a) Lemma 2 (b) Lemma 7.

between vertices x and x_{ee} which would both be visited before it) or the west (because $\overrightarrow{xx_n} \in H$). It also cannot enter it from the south via $\overrightarrow{x_{se}x_e}$ because H, which begins on the boundary of G in a quadrant other than $G_{ne}[x]$, would have to circle clockwise from x_n around to x_{se}. In doing so, it crosses x_e's vertical grid line L at a vertex above x_e before reaching x_{se} below it, thus leaving x_e unvisited between two visited vertices on L. This contradicts Lemma 1. Therefore, $\overrightarrow{x_{ne}x_e} \in H$.

By Lemma 1, at the time $\overrightarrow{x_{ne}x_e}$ is traversed, no vertex south of x_{ne} on its vertical grid line is visited. Also by Lemma 1, at the time x_{ne} is visited, no vertex east of it on its horizontal grid line is visited (because x_n is already visited). Therefore, x_{ne} is the first vertex visited by H in $G_{se}[x_{ne}]$. $\square$

Lemma 5 *Let $i \in \{n, s\}$ and $j \in \{e, w\}$. Let $x \in G$ be the first vertex visited by H in $G_{ij}[x]$ and $\overrightarrow{xx_j} \in H$ ($\overrightarrow{xx_i} \in H$). Then the two parallel grid lines containing x and x_j (x_i) in $G_{ij}[x]$ form a zigzag sequence in H.*

Proof. The case ($i = n$, $j = e$, $\overrightarrow{xx_n} \in H$) follows immediately from Lemma 4, by induction on the number of vertices to the right of x. The other cases are similar by symmetry of rotations and reflections. $\square$

Lemma 6 *If $\overrightarrow{vw}$ is the first edge visited in a zigzag sequence σ, the zigzag edges to each side of $\overrightarrow{vw}$ are visited in sequential order starting with $\overrightarrow{vw}$. (E.g., the edge adjacent to (furthest from) $\overrightarrow{vw}$ on a side is visited first (last) among all the edges on that side). If σ is separating, then (i) if the last visited edge in σ points north (west), then H ends in Γ_{top} (Γ_{bot}), and (ii) if the last visited edge in σ points south (east) then H does not end in Γ_{top} (Γ_{bot}).*

Proof. The first claim of this lemma follows immediately from Lemma 1. By Lemma 3, H must end in either a top corner vertex of Γ_{top}, a left corner vertex of Γ_{bot}, or the rightmost bottom corner of G. If the last visited edge in σ points north, there is no way for H to return to any of these end points other than the ones in Γ_{top}. The other claims follow from similar arguments. $\square$

Due to space considerations, the proof of the following lemma can be found in Appendix 7.

Lemma 7 *Let x be a vertex (interior or boundary) of G. If x is first visited by H, from among all vertices of $G_{ne}[x]$, then the following properties hold:*

(1) *If $\overrightarrow{xx_n} \in H$, then for each step $\Gamma[c] \subset G_{ne}[x]$ of odd height, either $h(x, \Gamma[c])$ is even, or c is the end point of H.*
(2) *If $\overrightarrow{xx_s} \in H$, then for each step $\Gamma[c] \subset G_{ne}[x]$ of odd height, either $h(x, \Gamma[c])$ is odd, or c is the end point of H.*

The following lemma follows immediately from Lemma 7 (by symmetry of rotations and reflections).

Lemma 8 *Let x be a vertex (interior or boundary) of G. If x is first visited by H, from among all vertices of $G_{sw}[x]$, then the following properties hold:*

(1) *If $\overrightarrow{xx_w} \in H$, then for each step $\Gamma[c] \subset G_{sw}[x]$ of odd width, either $v(x, \Gamma[c])$ is even, or c is the end point of H.*
(2) *If $\overrightarrow{xx_e} \in H$, then for each step $\Gamma[c] \subset G_{sw}[x]$ of odd width, either $v(x, \Gamma[c])$ is odd, or c is the end point of H.*

4 Existence of a Squeeze-Free Hamiltonian Path

An algorithm we call VisitWest(G) constitutes a key ingredient in our Hamiltonian path algorithm. It can be applied on any staircase subgraph G that satisfies the condition that, if x is the top right corner of G, then $v(x, \Gamma)$ is even for any step Γ of G of odd width. It constructs a squeeze-free Hamiltonian path H that starts at x and moves in the direction $\overrightarrow{xx_w}$. From x_w H follows *pattern*1 until it reaches a step of G at odd vertical distance from x, then it follows *pattern*2 until it reaches a step of G at even vertical distance from x, then repeats. See Figure 5a for an example and Appendix 8 for more details.

In building a Hamiltonian path for an arbitrary staircase graph G, we will use three other variations of the VisitWest algorithm on various subgraphs of G – namely VisitEast, VisitNorth and VisitSouth. The algorithm VisitEast is similar to VisitWest, with the only difference that the starting point is at the top left corner and H begins by moving east. One may view the path produced by VisitWest(G_{sw}[x]) as composed of the subpath extending from x to the horizontally opposite corner y (see Figure 5a), the edge $\overrightarrow{yy_s}$, and the path produced by VisitEast($G_{sw}[x_s]$). The precondition for VisitEast(G) is that $v(x, \Gamma)$ is odd for each step Γ of G of odd width.

The algorithm VisitNorth is identical to VisitWest, when operating on copy of G rotated clockwise by 90° and then reflected vertically. In this case, the first edge in H is $\overrightarrow{xx_n}$, where x is the lower left corner of G. The precondition for VisitNorth(G) is that $h(x, \Gamma)$ is even for each step Γ of G of odd height. The algorithm VisitSouth is similar to VisitNorth, with the only difference that the starting point is at the top left corner of the lowest stair and H begins by moving south. The precondition for VisitSouth(G) is that $h(x, \Gamma)$ is odd for each step Γ of G of odd height. Thus we have the following lemma.

Lemma 9 *Let G be a staircase graph that satisfies the preconditions of the* VisitWest*(*East, North, South*) algorithm. The path H produced by running* VisitWest*(*East, North, South*) on G is a squeeze-free Hamiltonian path for G.*

We now prove our main result in Theorems 10, 11, and 12. The proofs of Theorems 11 and 12 are similar to Theorem 10, so we leave their details for Appendix 9.

Theorem 10 *Let x be a vertex on a horizontal segment of ∂G that is not the top left corner of a step. There is a squeeze-free Hamiltonian path H that starts at x and includes $\overrightarrow{xx_w}$ if and only if the following three conditions hold:*

(1) *For each step Γ_e of odd height lying east of x, $h(x, \Gamma_e)$ is odd. If the width of $G_{se}[x]$ is odd, then Γ_{top} is exempt from this condition.*
(2) *For each step Γ_w of odd width lying west of x, $v(x, \Gamma_w)$ is even. If the height of $G_{se}[x]$ is even, then Γ_{bot} is exempt from this condition.*
(3) *If the height of $G_{se}[x]$ is even, then the width of $G_{se}[x]$ is also even.*

Proof. For the if direction, assume that the three conditions hold. Our goal is to find a squeeze-free Hamiltonian path H in G.

If-Case 1. Consider first the case where the height of $G_{se}[x]$ is odd. (See Figure 5b.) By condition (2), any step Γ_w of odd width lying west of x (including Γ_{bot}) satisfies the restriction that $v(x, \Gamma_w)$ is even. This implies that $G_{sw}[x]$ satisfies the precondition of the VisitWest algorithm, so we begin with $H =$VisitWest($G_{sw}[x]$). (See Figure 5c). By Lemma 9 all vertices of $G_{sw}[x]$ have been visited by this method.

Let z be the vertex at the intersection between the bottom boundary segment of G and the vertical line through x. Because the height of $G_{se}[x]$ is odd, H points east into z. We let H take another step east (so $\overrightarrow{zz_e} \in H$), then proceed depending on the parity of the width of $G_{se}[x]$. If the width of $G_{se}[x]$ is even, by condition (1) each step Γ_e lying east of z_e (including Γ_{top}) satisfies the restriction that $h(z_e, \Gamma_e)$ is even (because $h(z, \Gamma_e)$ is odd). This implies that $G_{ne}[z_e]$ satisfies the precondition of the VisitNorth algorithm, so we append to H the path produced by VisitNorth($G_{ne}[z_e]$).

The case where x is a reflex corner needs special attention, because in this case the left side xy of the step

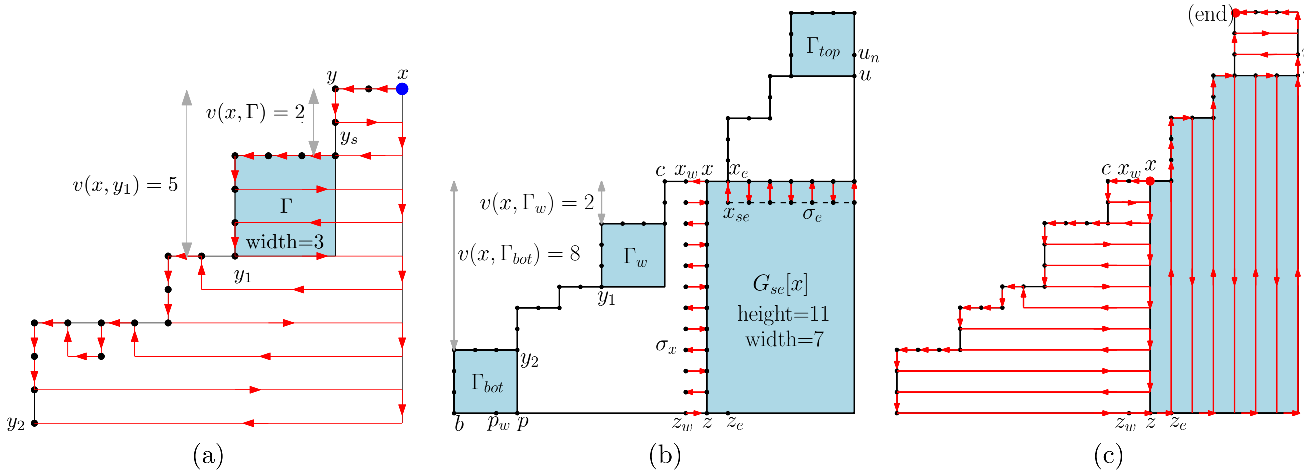


Figure 5: (a) VisitWest($G_{sw}[x]$). (b) Theorem 10: $G_{se}[x]$ has odd height and width. (c) Hamiltonian path.

$\Gamma[y]$ with lower left corner x is not in $G_{ne}[z_e]$. In this case $h(x, \Gamma[y]) = 0$ is even, and by condition (1) the height of $\Gamma[y]$ is even. This allows us to replace the vertical segment in H running alongside xy by a zigzag subpath of unit width that includes all vertices on the segment xy (similar to the *pattern*2 procedure). This along with Lemma 9 guarantees that, at the end of this procedure, all vertices of G have been visited.

If the width of $G_{se}[x]$ is odd (as in Figure 5c), condition (1) allows $h(x, \Gamma_{top})$ to be even and Γ_{top} to be of odd height. In this case $h(z_e, \Gamma_{top})$ is odd and $G_{ne}[z_e]$ does not satisfy the precondition imposed by VisitNorth. We handle this situation by restricting our attention to the subgraph $G'_{ne}[z_e]$ obtained from $G_{ne}[z_e]$ after eliminating Γ_{top}, with the exception of the lowest row of vertices in Γ_{top}. (The subgraph $G'_{ne}[z_e]$ is shaded in Figure 5c.) Note that $G'_{ne}[z_e]$ satisfies the precondition of VisitNorth, so we append to H the path produced by VisitNorth($G'_{ne}[z_e]$). By Lemma 9, at the end of this procedure all vertices of $G'_{ne}[z_e]$ have been visited. If x is a reflex corner, H absorbs the vertices along the vertical boundary segment sitting on x as described above. At this point, H is a squeeze-free Hamiltonian path for $G_{sw}[u]$, where u is the lower right corner u of Γ_{top}.

Because the width of $G_{se}[x]$ is odd, at the end of VisitNorth H points north into u. We add $\overrightarrow{uu_n}$ and $\overrightarrow{u_nu_{ne}}$ to H, then let H follow *pattern1* (reflected vertically) across Γ_{top} until all vertices of G have been visited. The result is a squeeze-free Hamiltonian path for G.

If-Case 2. The case when $G_{se}[x]$ has even height is similar and omitted for space considerations. See Appendix 9.

For the only-if direction, assume that there is a squeeze-free Hamiltonian path H in G. We next show that the three theorem conditions hold. We begin with the following two observations. Refer to Figure 5b.

(a) Because x is the start point of H and $\overrightarrow{xx_w} \in H$ (by the theorem statement), by Lemma 5 the two rightmost columns in $G_{sw}[x]$ form a separating zigzag sequence σ_x.

(b) By Lemma 1, $\overrightarrow{x_{ee}x_e} \notin H$. In addition, $\overrightarrow{x_{ne}x_e} \notin H$, because such an edge could only exist in H if x or x_e were a reflex corner, and in either case it would require that $H[x, x_{ne}]$ intersect a vertical line L through x_e both above and below x_e, which contradicts Lemma 1. Therefore $\overrightarrow{x_{se}x_e} \in H$. We claim that none of the vertices in $G_{ne}[x_{se}]$ has been visited at the time x_{se} is visited. Otherwise, if there were such a vertex $y \in G_{ne}[x_{se}]$ already visited at the time x_{se} is visited, then $H[x, y]$ would have to intersect the horizontal line L passing through x_{se} in two vertices on either side of x_{se}, leaving x_{se} unvisited between two visited vertices on L. This contradicts Lemma 1. Thus we are in the context of Lemma 5, with x_{se} being first visited among all vertices of $G_{ne}[x_{se}]$ and $\overrightarrow{x_{se}x_e} \in H$, therefore the two bottom rows of $G_{ne}[x_{se}]$ form a zigzag sequence σ_e.

Condition (1). By observation (b) above, x_{se} is the first visited by H among all vertices of $G_{ne}[x_{se}]$, and $\overrightarrow{x_{se}x_e} \in H$. Thus we can use the result of Lemma 7 on $G_{ne}[x_{se}]$ to show that, for each step Γ_e of odd height other than Γ_{top}, $h(x_{se}, \Gamma_e)$ is even and $h(x, \Gamma_e) = 1 + h(x_{se}, \Gamma_e)$ is odd. If the width of $G_{se}[x]$ is even, then the rightmost edge in σ_e points south. By Lemma 6, the end point of H lies outside of Γ_{top}. Thus Lemma 7 applies to show that, if the height of Γ_{top} is odd, then $h(x, \Gamma_{top})$ is odd. Thus condition (1) holds.

Condition (2). Because x is the start point of H and $\overrightarrow{xx_w} \in H$, we can use the result of Lemma 8 on $G_{sw}[x]$ to show that, for each step Γ_w of odd width other than Γ_{bot}, $v(x, \Gamma_w)$ is even. If the height of $G_{se}[x]$ is odd (see Figure 5b), then the lowest edge in σ_x points east.

By Lemma 6, the end point of H is not in Γ_{bot}. Thus Lemma 7 applies to show that, if the width of Γ_{bot} is odd, then $v(x, \Gamma_{bot})$ is even. Thus condition (2) holds.

Condition (3). Assume that the height of $G_{se}[x]$ is even. Because it is even, the lowest edge in σ_x points west. By Lemma 6, the end point of H lies in Γ_{bot}. This implies that the rightmost edge in σ_e points south (otherwise H would end in Γ_{top}). Because $\overrightarrow{x_{se}x_e} \in \sigma_e$ points north (see observation (b)), this is possible only if σ_e has odd length. This implies that $G_{se}[x]$ has even width, so condition (3) holds. This completes the proof. $\square$

Theorem 11 *Let $x \in \partial G$ be a vertex on a horizontal segment of a step $\Gamma[c]$ of G, such that removing xx_s does not disconnect G. Let $\Gamma[c_1]$ be the first odd width step that lies west of x (if one exists). There is a squeeze-free Hamiltonian path H that starts at x and includes $\overrightarrow{xx_s}$ if and only if the following conditions hold:*

(1) *For each step Γ_e of odd height lying east of x, $h(x, \Gamma_e)$ is odd. If the width of $G_{se}[x]$ is odd, then Γ_{top} is exempt from this restriction.*
(2) *If c_1 exists, then for each step Γ_w of odd width lying west of $\Gamma[c_1]$, $v(c_1, \Gamma_w)$ is even. If the height of $G_{se}[c_1]$ is even, then Γ_{bot} is exempt from this restriction.*
(3) *If c_1 exists and the height of $G_{se}[c_1]$ is even, then the width of $G_{se}[x]$ is also even.*
(4) *If x does not lie on Γ_{bot}, then $|xc|$ is even.*
(5) *If $|xc|$ is odd, then the width of $G_{se}[x]$ is even.*

Theorem 12 *Let $x \in \partial G$ be a vertex on a horizontal segment of a step $\Gamma[c]$ of G, such that removing xx_e does not disconnect G. Let $\Gamma[c_2]$ be the first odd height step that lies east of x (if one exists). There is a squeeze-free Hamiltonian path H that starts at x and includes $\overrightarrow{xx_e}$ if and only if the following conditions hold:*

(1) *For each step Γ_w of odd width lying west of x, $v(x, \Gamma_w)$ is odd. If the height of $G_{se}[x]$ is odd, then Γ_{bot} is exempt from this restriction.*
(2) *If c_2 exists, then for each step Γ_e of odd height lying east of $\Gamma[c_2]$, $h(c_2, \Gamma_e)$ is even. If the width of $G_{se}[c_2]$ is even, then Γ_{top} is exempt from this restriction.*
(3) *If c_2 exists and the width of $G_{se}[c_2]$ is even, then the height of $G_{se}[x]$ is also even.*
(4) *If x does not lie on Γ_{bot}, then $|xc|$ is even.*
(5) *If $|xc|$ is odd, then the height of $G_{se}[x]$ is odd.*
(6) *If c_2 exists and $h(x, c_2) = 0$, then $\Gamma[c_2] = \Gamma_{top}$, $|xc|$ is even, and the height of $G_{se}[x]$ is even.*

The case where the first edge in H is $\overrightarrow{xx_n}$ is symmetric to the case where the first edge in H is $\overrightarrow{xx_w}$ (subject to a 90° clockwise rotation and a vertical reflection). Similarly, the case where the start point x of H is on a vertical staircase segment is symmetric to the case where x is on a horizontal staircase segment.

5 Complexity of the Decision Problem

Using a sweep line algorithm described in Appendix 10, we have the following result.

Theorem 13 *Given a staircase grid graph G represented as a sequence of t pairs of numbers indicating the height and width of each step in order from left to right, there is an $O(t)$ algorithm that decides whether G admits a squeeze-free Hamiltonian path starting from a vertex on ∂G.*

6 Conclusions

In this paper we give an $O(t)$ algorithm for deciding if a staircase grid graph with t steps has a squeeze-free Hamiltonian path starting at a boundary vertex on a step. Although not included here, we can use the same proof techniques to determine similar necessary and sufficient conditions for the existence of such paths starting from the bottom or right side of the staircase. We conjecture though that if there exists a squeeze-free Hamiltonian path starting at the bottom or right side, then there also exists a squeeze-free Hamiltonian path starting from a vertex on a step.

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Appendix

7 Proof of Lemma 7 from Section 3

This section contains the proof of Lemma 7 that was omitted from the body of the paper due to space considerations. We begin with Lemma 14 which is referenced in Lemma 7.

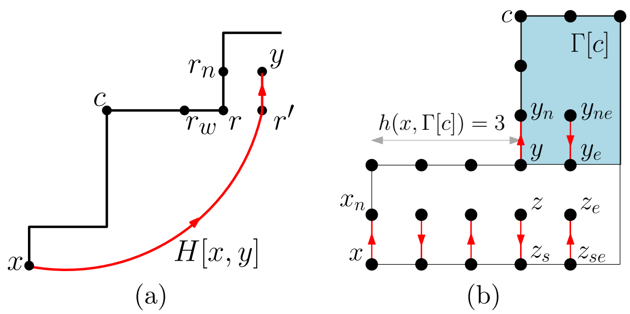


Figure 6: (a) Lemma 14. (b) Lemma 7

Lemma 14 *Let x be the start point of H. For any reflex corner vertex $r \in G_{ne}[x]$ ($r \in G_{sw}[x]$), r is first visited among all vertices in $G_{ne}[r]$ ($G_{sw}[r]$).*

Proof. Let $r \in G_{ne}[x]$ be an arbitrary reflex corner vertex, and let $\Gamma[c]$ be the step with top right corner vertex r. Because r is a reflex corner, r_n and r_w exist. No vertex $y \in \partial G$ above r can be visited before r by Lemma 2. Therefore r_n is visited after r. So assume for contradiction that there is some other vertex $y \in G_{ne}[r]$ that is visited by H prior to r. See Figure 6a. Then there is a vertex r' to the right of r, at the intersection between the horizontal through r and $H[x, y]$. By Lemma 1, r_w is not visited at the time r is visited (because r' is already visited). But $H[x, r]$, which links two boundary vertices, splits G into two pieces, and H cannot visit both unvisited vertices r_n (to the right of $H[x, r]$) and r_w (to the left of $H[x, r]$) without crossing itself. The arguments for the case $r \in G_{sw}[x]$ are symmetric. □

Lemma 7 *Let x be a vertex (interior or boundary) of G. If x is first visited by H, from among all vertices of $G_{ne}[x]$, then the following properties hold:*

(1) *If $\overrightarrow{xx_n} \in H$, then for each step $\Gamma[c] \subset G_{ne}[x]$ of odd height, either $h(x, \Gamma[c])$ is even, or c is the end point of H.*

(2) *If $\overrightarrow{xx_s} \in H$, then for each step $\Gamma[c] \subset G_{ne}[x]$ of odd height, either $h(x, \Gamma[c])$ is odd, or c is the end point of H.*

Proof. For (1), assume for contradiction that there exists a step $\Gamma[c] \subset G_{ne}[x]$ of odd height such that $h(x, \Gamma[c])$ is odd and c is not the end point of H. Refer to Figure 4b. Let y be the left bottom corner of $\Gamma[c]$. Consider the three possible directions from which H might visit y_n. Observe that $\overrightarrow{y_{nn}y_n} \notin H$, because if it were then subpath $H[x, y_{nn}]$ would have to intersect the vertical line L passing through y_n in two vertices on either side of y_n, leaving y_n unvisited between two visited vertices on L. This contradicts Lemma 1. So consider the case when $\overrightarrow{y_{ne}y_n} \in H$. Then y_{ne} is the first vertex visited in $G_{nw}[y_{ne}]$ (otherwise, if there were a vertex $y' \in G_{nw}[y_{ne}]$ visited prior to y_{ne}, then the subpath $H[x, y']$ would cross the vertical through y_{ne} at two vertices on either side of y_{ne}, contradicting Lemma 1). Thus we can apply Lemma 5 to show that $\overrightarrow{y_{ne}y_n}$ is the start of a zigzag sequence in $G_{nw}[y_{ne}]$, and because $|yc|$ is odd, the last edge in the zigzag is directed into corner c, thus H ends at c. This contradicts our assumption that H does not end at c, and thus $\overrightarrow{y_{ne}y_n} \notin H$.

Before completing the proof of property (1), first observe that by Lemma 5, the bottom two rows of $G_{ne}[x]$ form a zigzag sequence. Refer to Figure 6b. Because $h(x, \Gamma[c])$ is odd and the zigzag starts with the upward directed edge $\overrightarrow{xx_n}$, the zigzag edge in c's vertical grid line points downward. Label this edge $\overrightarrow{zz_s}$. Now consider the last case for property (1) which is when $\overrightarrow{yy_n} \in H$. By Lemma 14, y is first visited among all vertices in $G_{ne}[y]$. By Lemma 5, $\overrightarrow{y_{ne}y_e} \in H$, which points in a direction opposite to that of $\overrightarrow{z_{se}z_e}$, contradicting Lemma 1.

For property (2), note that because $\overrightarrow{xx_s} \in H$, no vertex below x on its vertical line is visited before x. This combined with the fact that x is the first vertex visited in $G_{ne}[x]$ shows that x is also the first vertex visited in $G_{se}[x]$. Thus by Lemma 5, the upper two rows of $G_{se}[x]$ form a zigzag sequence, with the first edge, $\overrightarrow{xx_s}$, pointing downward. The rest of the proof is analogous to the proof of property (1). □

8 Additional Details For VisitWest from Section 4

Algorithm 1 details the VisitWest algorithm. See Figure 5a for an example.

Algorithm 1: VisitWest(**staircase** G)

Precondition: Let t be the upper right corner of G. For each step Γ of G of odd width, $v(t, \Gamma)$ is even.

Initialize $x \leftarrow t$ and $H \leftarrow \{\overrightarrow{xx_w}\}$. Let b be the lower left corner of G.

repeat

    Let y_1 be the reflex corner west of x closest to x, such that $v(x, y_1)$ is odd.

    If no such vertex exists, then $y_1 \leftarrow b$.

    Let y_2 be the reflex corner west of y_1 closest to y_1, such that $v(y_1, y_2)$ is odd.

    If no such vertex exists, then $y_2 \leftarrow b$.

    From x, let H follow *pattern*1 south-west until it meets y_1.

    From y_1, let H follow *pattern*2 south-west until it meets y_2.

    Reset $x \leftarrow y_2$.

until *all vertices of G have been visited*;

Output H.

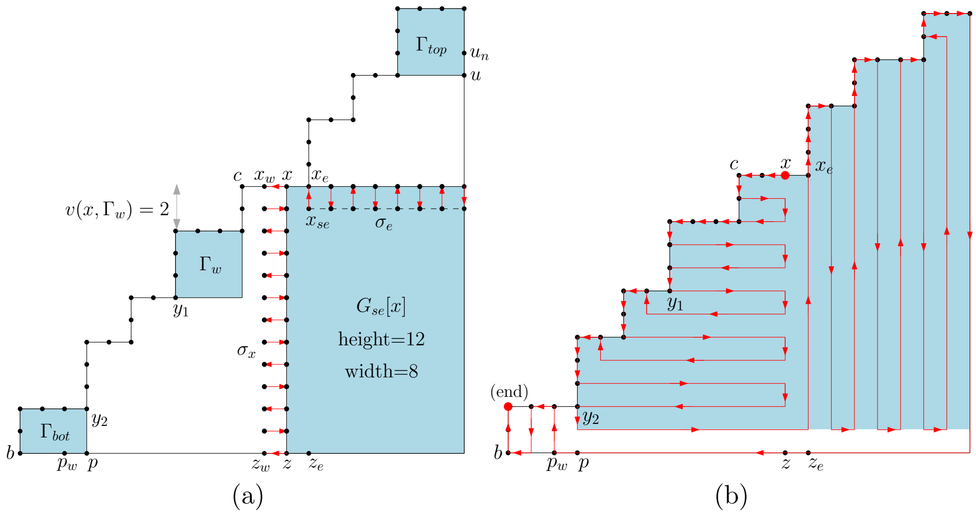


Figure 7: (a) Theorem 10: $G_{se}[x]$ has even height and even width. (b) Hamiltonian path.

9 Proofs of Theorems 10,11, and 12 from Section 4

We begin by supplying the portion of the proof of Theorem 10 that was omitted in the body of the text.

Omitted Portion of Theorem 10 proof: If-Case 2. Consider now the case where the height of $G_{se}[x]$ is even, as depicted in Figure 7a. By condition (3), the width of $G_{se}[x]$ is also even. Let G' be the graph obtained from G after eliminating all vertices on the lowest boundary segment of G, along with all vertices in Γ_{bot}, with the exception of those lying on the right boundary segment of Γ_{bot}. (G' is shown shaded in Figure 7b.) Thus $G'_{se}[x]$ is of odd height and even width. We trace H across $G'_{se}[x]$ using the same procedure as described above for the case where $G_{se}[x]$ was of odd height and even width. It can be verified that, at the end of this procedure, H points south into the lower right corner of G'. At this point H takes a unit step south, then continues west along the bottom boundary segment of G up to the vertex p_w, where p is the lower right corner of Γ_{bot}. From p_w H follows *pattern1* (rotated counterclockwise by 90° and reflected vertically) until all vertices of Γ_{bot} have been visited. At that point, all vertices of G have been visited, and H is a squeeze-free Hamiltonian path for G.

We now provide complete proofs of Theorems 11 and 12.

Theorem 11 *Let $x \in \partial G$ be a vertex on a horizontal segment of a step $\Gamma[c]$ of G, such that removing xx_s does not disconnect G. Let $\Gamma[c_1]$ be the step of odd width closest to x that lies west of x (if one exists). There is a squeeze-free Hamiltonian path H that starts at x and includes $\overrightarrow{xx_s}$ if and only if the following five conditions hold:*

(1) *For each step Γ_e of odd height lying east of x, the horizontal distance $h(x, \Gamma_e)$ is odd. If the width of $G_{se}[x]$ is odd, then Γ_{top} is exempt from this restriction.*

(2) *If c_1 exists, then for each step Γ_w of odd width lying west of $\Gamma[c_1]$, the vertical distance $v(c_1, \Gamma_w)$ is even. If the height of $G_{se}[c_1]$ is even, then Γ_{bot} is exempt from this restriction.*

(3) *If c_1 exists and the height of $G_{se}[c_1]$ is even, then the width of $G_{se}[x]$ is also even.*

(4) *If x does not lie on Γ_{bot}, then $|xc|$ is even.*

(5) *If $|xc|$ is odd, then the width of $G_{se}[x]$ is even.*

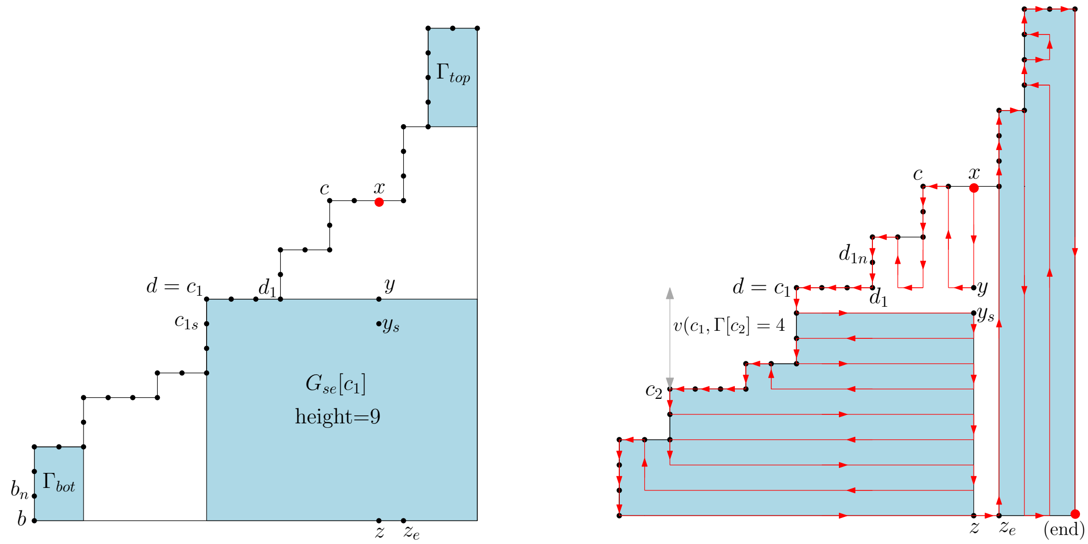
Proof. We consider each of the two directions (if, and only if) in turn.

If direction. For the if direction, assume that the five conditions hold. Our goal is to find a squeeze-free Hamiltonian path H in G. Let b be the lower left corner of Γ_{bot}.

If-Case 1. Consider first the case where either of the following is true: (i) c_1 does not exist and $|xc|$ is even, and (ii) c_1 exists and the height of $G_{se}[c_1]$ is odd. In the latter case x may not lie on Γ_{bot} (due to the existence of c_1), and by condition (4) $|xc|$ is even.

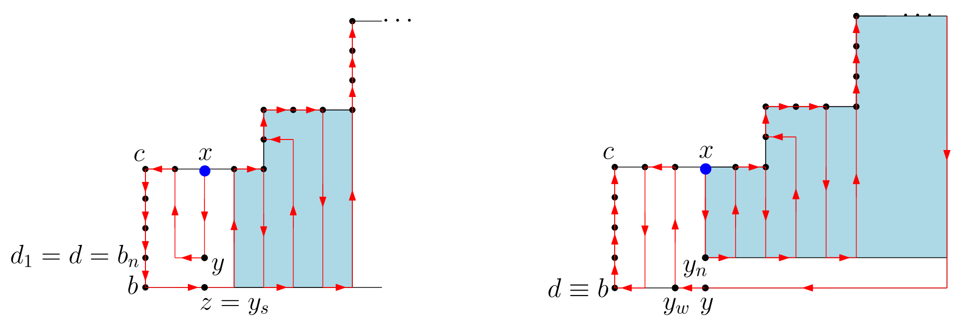
Define $d = c_1$ if c_1 exists (see Figure 8), and $d = b_n$ otherwise (see Figure 9a). Let y be the intersection point between the vertical through x and the horizontal through d. In either case, y_s exists. (Note that in case (i) when x is the top left corner of $\Gamma[bot]$, $G_{nw}[y]$ degenerates to a vertical line segment.) Otherwise, by the definition of c_1 and the fact that $|xc|$ is even, every step in $G_{nw}[y]$ has even width. In either case *pattern*1 (starting with $\overrightarrow{xx_s}$) can be used to visit all vertices of $G_{nw}[y]$ (regardless of the existence of c_1). We let H follow this path until it reaches the lower left corner d_1 of $G_{nw}[y]$, coming from north (so $\overrightarrow{d_{1n}d_1} \in H$). If d_{1w} exists, H continues west as far as it can go (up to c_1 in Figure 8b). Next H takes a step south.

Let z be the intersection point between the vertical through x and the bottom boundary segment of G. If c_1 does not exist, $z = y_s$ and H continues east up to z_e (see Figure 9a). If c_1 exists, H visits all vertices of $G_{sw}[y_s]$ on its way to z_e as follows. By condition (2), any step Γ_w of odd width lying west of c_1 (including Γ_{bot}) satisfies the restriction that $v(c_1, \Gamma_w)$ is even. This implies that $v(c_{1s}, \Gamma_w)$ is odd, so the precondition of VisitEast restricted to the subgraph



(a) (b)

Figure 8: Theorem 11: (a) $G_{se}[c_1]$ has odd height (b) Hamiltonian path H.



(a) (b)

Figure 9: Theorem 11, $x \in \Gamma_{bot}$ (a) $|xc|$ even (b) $|xc|$ odd.

$G_{sw}[y_s]$ (see left shaded subgraph in Figure 8b) is satisfied. We append to H the path produced by VisitEast($G_{sw}[y_s]$). Using the result of Lemma 9, it can be verified that at this point H is a squeeze-free Hamiltonian path in $G_{nw}[z]$. Let H take another step $\overrightarrow{zz_e}$.

If the width of $G_{se}[x]$ is even, or if Γ_{top} has even height, then by condition (1) $h(x, \Gamma_e)$ is odd for every step Γ_e of odd height lying east of x. Then $h(z_e, \Gamma_e)$ is even and therefore the precondition of VisitNorth is satisfied when restricted to $G_{ne}[z_e]$ (see right shaded subgraph in Figure 8b). We append to H the path produced by VisitNorth($G_{ne}[z_e]$).

The case where x is a reflex corner needs special attention, because in this case the left side xy of the step $\Gamma[y]$ with lower left corner x is not in $G_{ne}[z_e]$. In this case $h(x, \Gamma[y]) = 0$ is even, and by condition (1) the height of $\Gamma[y]$ is even. Then we can replace the vertical segment in H running alongside xy by a zigzag subpath of unit width that includes all vertices on the segment xy (similar to the *pattern*2 procedure). This along with Lemma 9 guarantees that, at the end of this procedure, all vertices of G have been visited.

Finally, consider the situation where the width of $G_{se}[x]$ is odd and Γ_{top} has odd height. Let G' be the subgraph of G obtained by removing the top row of vertices in Γ_{top}. Then the top step in G' has even height. This along with condition (1) shows that $G'_{ne}[z_e]$ satisfies the precondition of VisitNorth, so we let H follow the path produced by VisitNorth($G'_{ne}[z_e]$). If x is a reflex corner of G, we adjust H as described above so that it visits all vertices on the vertical boundary segment incident on x. This along with Lemma 9 guarantees that H is a squeeze-free Hamiltonian path for G'. Because $G_{se}[x]$ has odd width, H ends up pointing north into top right corner of G'. Let H take another step north, then west all the way to the top left corner of Γ_{top}. The resulting path is a squeeze-free Hamiltonian path for G.

If-Case 2. Consider now the case where either of the following is true: (i) c_1 does not exist and $|xc|$ is odd, and (ii) c_1 exists and the height of $G_{se}[c_1]$ is even. In either case, conditions (3) and (5) guarantee that the width of $G_{se}[x]$ is even. If $|xc|$ is odd, by condition (4) x lies on Γ_{bot}.

Define $d = c_1$ if c_1 exists (see Figure 10), otherwise $d = b$ (see Figure 9b). Let y be the intersection point between the vertical through x and the horizontal through d. By

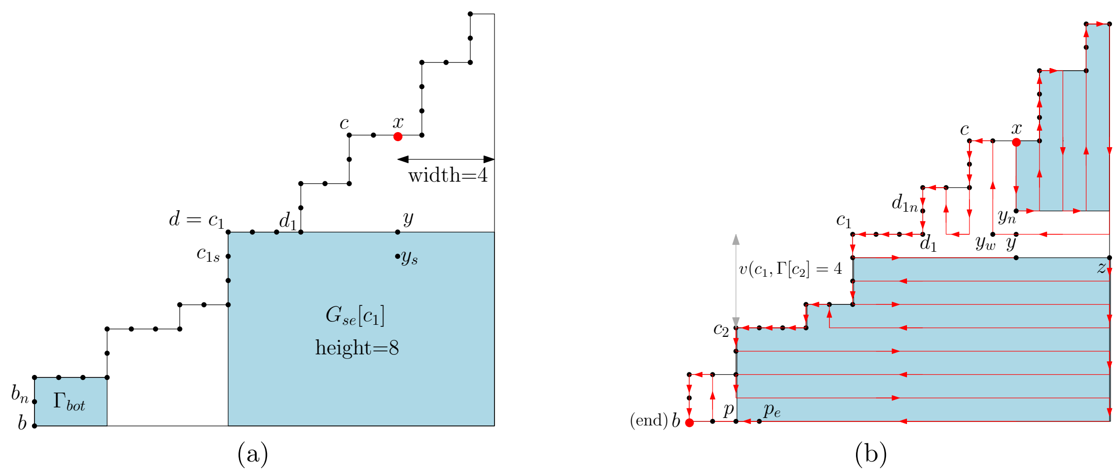


Figure 10: Theorem 11: (a) $G_{se}[c_1]$ has even height (b) Hamiltonian path H.

condition (1), for any step Γ_e of odd height lying east of x (including Γ_{top}), $h(x, \Gamma_e)$ is odd. Thus the precondition of VISITSOUTH restricted to the subgraph $G_{ne}[y_n]$ (see top shaded subgraph in Figures 10b and 9b) is satisfied. Let H follow the path produced by VISITSOUTH($G_{ne}[y_n]$). By Theorem 9, H is a squeeze-free Hamiltonian path for $G_{ne}[y_n]$. Because the width of $G_{se}[x]$ is even, H ends up pointing south into the lower right corner of $G_{ne}[y_n]$. Let H take another step south, then west all the way to y_w. As in the previous case, from y_w the path H follows *pattern*1 (starting with $\overrightarrow{y_w y_{nw}}$) restricted to $G_{nw}[y_w]$. If x lies on Γ_{bot}, *pattern*1 completes a squeeze-free Hamiltonian path for G (see Figure 9b).

If x does not lie on Γ_{bot}, the width of $G_{nw}[y_w]$ is even and therefore H arrives at the top right corner d_1 of $\Gamma[c_1]$ coming from north. Refer to Figure 10b. At this point H is a squeeze-free Hamiltonian path of $G_{ne}[d_1]$. From d_1 H continues west until it reaches c_1, then takes a step south. If $\Gamma[c_1]$ is identical to Γ_{bot}, then VISITEAST($G_{se}[c_{1s}]$) completes a squeeze-free Hamiltonian path for G.

Assume now that $\Gamma[c_1]$ is not identical to Γ_{bot}. Let G' be the subgraph of G obtained by removing the vertices in Γ_{bot}, with the exception of the rightmost vertex column in Γ_{bot}. Let z be the intersection point between the horizontal through c_{1s} and the right boundary segment of G. From c_{1s}, H follows the path produced by VISITEAST($G'_{sw}[z]$), up to the lower right corner p of Γ_{bot}. Condition (2) guarantees that the precondition of VISITEAST is satisfied, so at this point H is a squeeze-free Hamiltonian path of $G_{ne}[p]$ (by Lemma 9). Because the height of $G'_{sw}[z]$ is odd, H arrives at p from east, so $\overrightarrow{p_e p} \in H$. We let H take another step east, then trace *pattern*1 (rotated counterclockwise by 90° and reflected vertically) across Γ_{bot} to complete a squeeze-free Hamiltonian path for G.

Only if direction. For the only-if direction, assume that there is a squeeze-free Hamiltonian path H in G. We next show that the five lemma conditions hold. We begin with two observations:

(a) Because x is the starting point of H and $\overrightarrow{xx_s} \in H$ (by the theorem statement), we can use Lemma 5 to show that the top two rows in $G_s[x]$ form a separating zigzag sequence σ_x (see Figure 11).

(b) Assuming that c_1 exists, let d be the top right corner of $\Gamma[c_1]$. By Lemma 14 d is first visited among all vertices in $G_{sw}[d]$. If $\overrightarrow{dd_w} \in H$, by Lemma 5 the rightmost two columns of $G_{sw}[d]$ form a separating zigzag sequence σ_d (see Figure 11a).

If $\overrightarrow{dd_w} \notin H$, then $\overrightarrow{d_{sw} d_w} \in H$ (as the only way to reach d_w). Also note that d_{sw} is first visited among all vertices in $G_{nw}[d_{sw}]$. Otherwise, if there were a vertex $y \in G_{nw}[d_{sw}]$ already visited at the time d_{sw} is visited, then $H[x, y]$ would have to intersect the horizontal line L passing through d_{sw} in two vertices on either side of d_{sw}, leaving d_{sw} unvisited between two visited vertices on L. This contradicts Lemma 1. By Lemma 5, the two rows of $G_{nw}[d_{sw}]$ form a zigzag sequence σ_w (see Figure 11b). Because the width of $\Gamma[c_1]$ is odd (by definition), the leftmost edge in σ_w is $\overrightarrow{c_{1s} c_1}$, therefore c_1 is the end point of H.

Condition (1). Let $\Gamma_e = \Gamma[c_e]$ be an arbitrary step of odd height lying east of x (if no such step exists, there is nothing to prove). By Lemma 7, either $h(x, \Gamma_e)$ is odd or c_e is the end point of H. If $\Gamma_e \neq \Gamma_{top}$, then by Lemma 3 c_e cannot be the end point of H, so condition (1) holds. Assume now that $\Gamma_e = \Gamma_{top}$ and the width of $G_{se}[x]$ is even. In this case the rightmost edge in σ_x points south, and by Lemma 6 H many not end in Γ_{top}. Thus Lemma 7 applies again to show that $h(x, \Gamma_e)$ is odd, so condition (1) holds.

Condition (2). Assume that c_1 exists and let $\Gamma_w = \Gamma[c_w]$ be an arbitrary step of odd width lying west of c_1 (if no such step exists, there is nothing to prove). Note that $\Gamma[c_1] \neq \Gamma_{top}$ because of the existence of $\Gamma[c]$, and there is nothing to prove if $\Gamma[c_1] = \Gamma_{bot}$. By Lemma 3, H may not end at c_1. This along with observation (b) above implies that $\overrightarrow{dd_w} \in H$ (otherwise H would end at c_1). By Lemma 8 either $v(c_1, \Gamma_w)$ is even or c_w is the end point of H. If $\Gamma_w \neq \Gamma_{bot}$, then by Lemma 3 c_w cannot be the endpoint of H, so condition (2) holds. Assume now that $\Gamma_w = \Gamma_{bot}$ and that the height of

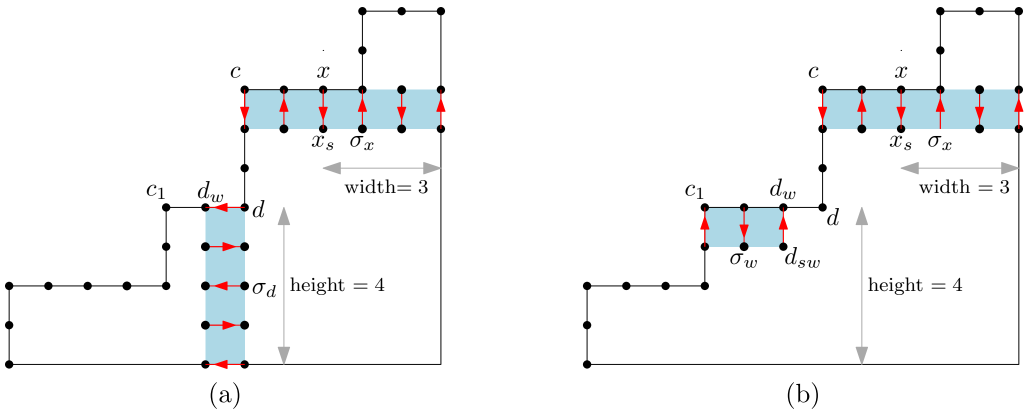


Figure 11: Theorem 11, only-if direction (a) $\overrightarrow{dd_w} \in H$ (b) $\overrightarrow{dd_w} \notin H$

$G_{se}[c_1]$ is odd. In this case the lowest edge in σ_x points east, and by Lemma 6 H cannot end in Γ_{bot}. This implies that c_w is not the endpoint of H, so condition (2) holds.

Condition (3). Assume that c_1 exists and the height of $G_{se}[c_1]$ is even. Note that $\Gamma[c_1] \neq \Gamma_{top}$ due to the existence of $\Gamma[c]$. We prove by contradiction that the width of $G_{se}[x]$ is even. Assume to the contrary that the width of $G_{se}[x]$ is odd. Then the rightmost edge in σ_x points north (see Figure 11a), and by Theorem 6 H must end in Γ_{top}. This along with observation (b) above implies that $\overrightarrow{dd_w} \in H$ (otherwise H would end at c_1). Because $\overrightarrow{dd_w} \in H$ and the height of $G_{se}[c_1]$ is even, the lowest edge in σ_d points west. By Lemma 6 H must end in Γ_{bot}, a contradiction. We conclude that the width of $G_{se}[x]$ is even.

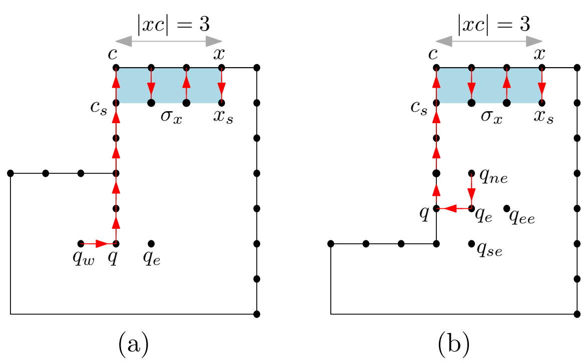


Figure 12: Theorem 11, only-if direction, $x \in \Gamma_{top}$ (a) q_w exists (b) q_w does not exist.

Condition (4). Assume that $x \notin \Gamma_{bot}$. We prove by contradiction that $|xc|$ is even. Assume to the contrary that $|xc|$ is odd. In this case $\overrightarrow{c_sc} \in H$ is the leftmost edge in σ_x, so H ends at c. Because $\Gamma[c] \neq \Gamma_{bot}$, Lemma 3 implies that c is the top left corner of Γ_{top}.

Let $q \in G$ be such that qc is the longest vertical subpath of H that ends at c. Refer to Figure 12. Then either $\overrightarrow{q_wq} \in H$ or $\overrightarrow{q_eq} \in H$ (as the only way to reach q). Note that $\overrightarrow{q_wq}$ would create a squeeze at q, because H must have visited q_e prior to q. Thus $\overrightarrow{q_wq} \notin H$ and therefore $\overrightarrow{q_eq} \in H$. This implies that q_w does not exist (otherwise H would create a squeeze at q). Because G has at least two steps (by our problem statement) we conclude that q lies above the lower left corner of Γ_{top}, and q_{se} exists. Note that all three vertices q_{se}, q_{ne} and q_{ee} must have been visited prior to q_e, so any of $\overrightarrow{q_{se}q_e}$, $\overrightarrow{q_{ne}q_e}$ and $\overrightarrow{q_{ee}q_e}$ would create a squeeze of q_e. This means that H has no way of reaching q_e, contradicting the fact that H is Hamiltonian. We conclude that $|xc|$ is even.

Condition (5). Assume that $|xc|$ is odd. By condition (4), $x \in \Gamma_{bot}$. In this case $\overrightarrow{c_sc} \in \sigma_x$, so H ends at c. If the width of $G_{se}[x]$ is odd, then the rightmost edge in σ_x points north, and by Lemma 6 H ends in Γ_{top}, a contradiction. Thus the width of $G_{se}[x]$ is even and condition (5) holds. This completes the proof. $\square$

Theorem 12 *Let $x \in \partial G$ be a vertex on a horizontal segment of a step $\Gamma[c]$ of G, such that removing xx_e does not disconnect G. Let $\Gamma[c_2]$ be the step of odd height closest to x that lies east of x (if one exists). There is a squeeze-free Hamiltonian path H that starts at x and includes $\overrightarrow{xx_e}$ if and only if the following six conditions hold:*

1. *For each step Γ_w of odd width lying west of x, the vertical distance $v(x, \Gamma_w)$ is odd. If the height of $G_{se}[x]$ is odd, then Γ_{bot} is exempt from this restriction.*
2. *If c_2 exists, then for each step Γ_e of odd height lying east of $\Gamma[c_2]$, the horizontal distance $h(c_2, \Gamma_e)$ is even. If the width of $G_{se}[c_2]$ is even, then Γ_{top} is exempt from this restriction.*
3. *If c_2 exists and the width of $G_{se}[c_2]$ is even, then the height of $G_{se}[x]$ is also even.*
4. *If x does not lie on Γ_{bot}, then $|xc|$ is even.*
5. *If $|xc|$ is odd, then the height of $G_{se}[x]$ is odd.*
6. *If c_2 exists and $h(x, c_2) = 0$, then $\Gamma[c_2] = \Gamma_{top}$, $|xc|$ is even, and the height of $G_{se}[x]$ is even.*

Proof. We consider each of the two directions (if, and only if) in turn.

If direction. For the if direction, assume that the five conditions hold. Our goal is to find a squeeze-free Hamiltonian path H in G. Let z be the intersection point between the horizontal through x and the right boundary segment of G.

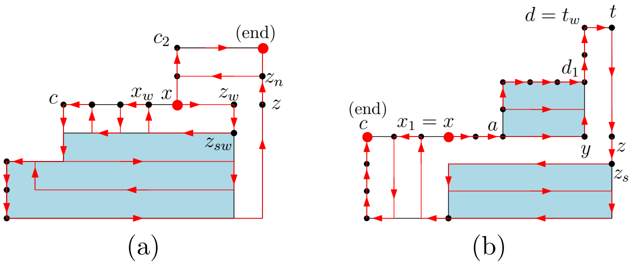


Figure 13: Theorem 12 (a) $h(x, c_2) = 0$ (b) c_2 does not exist and $x \in \Gamma_{bot}$.

If-Case 1. We begin with the simplest case where c_2 exists and $h(x, c_2) = 0$. Refer to Figure 13a. By condition (6) $|xc|$ is even and the height of $G_{se}[x]$ is even. In this case H proceeds east until it reaches z_w, then takes one step south. By condition (1) $v(x, \Gamma_w)$ is odd for every step Γ_w of odd width lying west of x. Thus $v(z_{sw}, \Gamma_w)$ is even and therefore the precondition of VisitEast is satisfied when restricted to $G_{sw}[z_{sw}]$ (see shaded subgraph in Figure 13a). We append to H the path produced by VisitEast($G_{sw}[z_{sw}]$). If x_w exists, we replace the segment in H running along x_wc by a zigzag subpath of unit height (starting with $\overrightarrow{x_{sw}x_w}$) that includes all vertices on x_wc. Because $|xc|$ is even, this subpath ends with $\overrightarrow{cc_s}$ and attaches seamlessly to the subpath of H starting at c_s. This along with Lemma 9 guarantees that, at this point, all vertices of $G_{sw}[z_w]$ have been visited. Because the height of $G_{se}[x]$ is even, H ends pointing east into the lower right corner of $G_{sw}[z_w]$. We let H take another step west to meet the the right boundary segment of G, then north up to z_n. Because $h(x, c_2) = 0$, by condition (6) $\Gamma[c_2] = \Gamma_{top}$, so we let H follow *pattern*1 (starting with $\overrightarrow{z_nz_{nw}}$) to complete a squeeze-free Hamiltonian path for G.

If-Case 2. Consider now the case where either c_2 does not exist, or c_2 exists and $h(x, c_2) > 0$. Define the following points: b be the lower left corner of G; t is the upper right corner of G; $d = d_1 = t_w$ if c_2 does not exist (see Figure 13b), else $d = c_2$ and d_1 is the lower reflex corner of $\Gamma[c_2]$ (see Figure 14a); and y is the intersection point between the horizontal through x and the vertical through d.

If x is not a reflex corner, H proceeds east from x until it reaches the first reflex vertex a. By the definition of c_2, every step in $G_{nw}[y]$ has even height See the top left shaded subgraph $G_{nw}[y]$ in Figure 14a for an example. This implies that *pattern*1 (starting with $\overrightarrow{aa_e}$) can be used to visit all vertices of $G_{nw}[y]$ (regardless of the existence of c_2). We let H follow this path until it reaches d_1 coming from the west (so $\overrightarrow{d_{1w}d_1} \in H$). If d_{1n} exists, H continues north as far as it can go (up to d). Next H takes a step east, then continues south all the way down to y_e. From here the path taken by H depends on the parity of the width of $G_{se}[c_2]$.

If-Case 2a. Assume first that the width of $G_{se}[c_2]$ is odd. Note that if c_2 does not exist, y_e coincides with z and H points south into z (see Figure 13b). Otherwise, if y_e does not coincide with z, then y_{ee} exists and is not on the boundary of G, because the width of $G_{se}[c_2]$ is odd. In this case H visits all vertices of $G_{ne}[y_{ee}]$ on its way to z as follows. By condition (2), any step Γ_e of odd height lying east of c_2 (including Γ_{top}) satisfies the restriction that $h(c_2\Gamma_e)$ is even. This implies that $h(y_{ee}, \Gamma_e)$ is also even, so the precondition of VisitNorth restricted to the subgraph $G_{ne}[y_{ee}]$ (see top right shaded subgraph in Figure 14a) is satisfied. We append to H the path produced by VisitNorth($G_{ne}[y_{ee}]$). Using the result of Lemma 9, it can be verified that at this point H is a squeeze-free Hamiltonian path in $G_{nw}[z]$. Because the width of $G_{se}[c_2]$ is odd (by our assumption), H arrives at z from north, so $\overrightarrow{z_nz} \in H$. Let H take another step $\overrightarrow{zz_s}$.

If the height of $G_{se}[x]$ is even, H continues along the path produced by VisitEast($G_{sw}[z_s]$). Arguments similar to the ones used in the first case show that at the end of this visit, H is a squeeze-free Hamiltonian path for G.

Assume now that the height of $G_{se}[x]$ is odd, as depicted in Figure 14a. Let $x_1 = x$ if x is on Γ_{bot} (see Figure 13b), otherwise x_1 is the top right corner of Γ_{bot} (see Figure 14a). Let G' be the subgraph obtained by removing from G all vertices left of the vertical through x_1. Condition (1) guarantees that, for each step Γ_w of odd width lying west of x, $v(x, \Gamma_w)$ is odd. This implies that $G'_{sw}[z_s]$ satisfies the precondition of VisitEast, so we let H follow the path produced by VisitEast($G'_{sw}[z_s]$). If x is not on Γ_{bot}, by condition (4) $|xc|$ is even. In this case we adjust H to incorporate all vertices on the horizontal boundary segment x_wc (if such vertices exist), as in the first case discussed above. At this point H is a squeeze-free Hamiltonian path for G'. Because $G_{se}[x]$ has odd height, H ends up pointing west into the lower left corner of G'. Let H take another step west, then follow *pattern*1 across the vertices in $G \setminus G'$. The resulting path H is a squeeze-free Hamiltonian path for G.

If-Case 2b. Assume now that the width of $G_{se}[c_2]$ is even. Refer to Figure 14b. By condition (3) the height of $G_{se}[x]$ is also even. By condition (5) $|xc|$ is even. By condition (1), for each step Γ_w of odd width lying west of x (including Γ_{bot}), $v(x, \Gamma_w)$ is odd. This implies that $v(y_{se}, \Gamma_w)$ is even, so the precondition of VisitWest restricted to the subgraph $G_{sw}[y_{se}]$ (see lower left shaded subgraph in Figure 14b) is satisfied. Let H follow the path produced by VisitWest($G_{sw}[y_{se}]$). If $|xc| > 0$, we adjust H to incorporate all vertices on the horizontal boundary segment x_wc, as discussed above. Because $|xc|$ is even, this adjustment is possible. Because the height of $G_{se}[x]$ is even, H ends up pointing east into the lower right corner u of $G_{sw}[y_e]$. At this point H is a squeeze-free Hamiltonian path for $G_{nw}[u]$. Let H take another step east, so $\overrightarrow{uu_e} \in H$.

Let G' be the graph obtained by removing from G all vertices above the bottom row in Γ_{top}. By condition (2), for each step Γ_e of odd height in G' lying east of c_2, $h(c_2, \Gamma_e)$ is even. Then $h(u_e, \Gamma_e)$ is also even. This implies that the precondition of VisitNorth restricted to the subgraph $G'_{ne}[u_e]$ (see right shaded subgraph in Figure 14b) is satisfied. Let H follow the path produced by VisitNorth($G'_{ne}[u_e]$). Be-

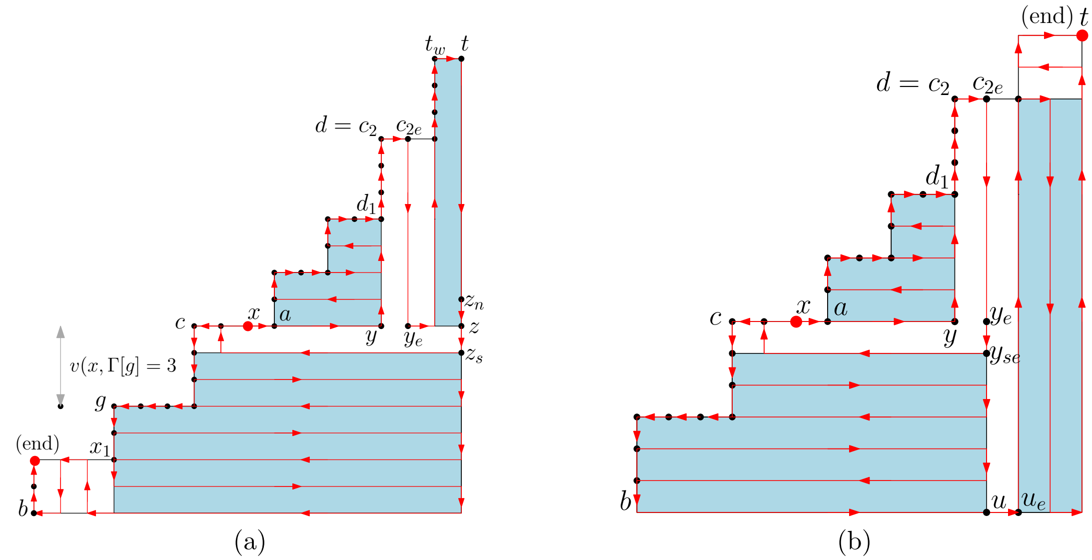


Figure 14: Path H from Theorem 12; the width of $G_{se}[c_2]$ is (a) odd (b) even.

cause the width of $G_{se}[c_2]$ is even, the width of $G'_{ne}[u_e]$ is also even, therefore H ends up pointing north into the lower right corner of Γ_{top}. Let H take another step north, then follow *pattern*1 across the vertices in $G \setminus G'$. The resulting path H is a squeeze-free Hamiltonian path for G.

Only if direction. For the only-if direction, assume that there is a squeeze-free Hamiltonian path H in G. We next show that the six theorem conditions hold. We begin with three observations:

(a) Because x is the starting point of H and $\overrightarrow{xx_e} \in H$ (by the theorem statement), we can use Lemma 5 to show that the two leftmost columns in $G_e[x]$ form a separating zigzag sequence σ_x. Refer to Figure 15.

(b) If x_w exists, then $\overrightarrow{x_{sw}x_w} \in H$. This is because $\overrightarrow{xx_e} \in H$ (by the theorem statement), and by Lemma 1 H cannot arrive at x_w from the left, therefore it must reach it coming from south. Also note that x_{sw} is first visited among all vertices in $G_{nw}[x_{sw}]$. Otherwise, if there were a vertex $y \in G_{nw}[x_{sw}]$ already visited at the time x_{sw} is visited, then $H[x, y]$ would have to intersect the horizontal line L passing through x_{sw} in two vertices on either side of x_{sw}, leaving x_{sw} unvisited between two visited vertices on L. This contradicts Lemma 1. By Lemma 5 the two rows in $G_{nw}[x_{sw}]$ form a zigzag sequence σ_w (provided that x_w exists).

(c) If c_2 exists, let d be the lower left corner of $\Gamma[c_2]$. By Lemma 14 d is first visited among all vertices in $G_{ne}[d]$. If $\overrightarrow{dd_n} \in H$, by Lemma 5 the bottom two rows of $G_{ne}[d]$ form a separating zigzag sequence σ_d (see Figure 15a). If $\overrightarrow{dd_n} \notin H$, then $\overrightarrow{d_{ne}d_n} \in H$ (as the only way to reach d_n). Arguments similar to the ones used in observation (b) above show that d_{ne} is first visited among all vertices in $G_{nw}[d_{ne}]$. By Lemma 5, the two columns of $G_{nw}[d_{ne}]$ form a zigzag sequence σ_n (see Figure 15b). Because $\Gamma[c_2]$ is of odd height (by definition), the topmost edge in σ_n is $\overrightarrow{c_{2e}c_2}$, therefore c_2 is the end point of H.

Condition (1). Because $\overrightarrow{xx_e} \in H$ and x is the start point of H, we can use the result of Lemma 8 to show that, for each step Γ_w of odd width other than Γ_{bot} lying west of x, $v(x, \Gamma_w)$ is odd. If the height of $G_{se}[x]$ is even, then the lowest edge in σ_x points east. By Lemma 6, H does not end in Γ_{bot}. Thus Lemma 8 applies to show that, if the height of $G_{se}[x]$ is even, then $v(x, \Gamma_{bot})$ is odd.

Condition (2). Assume that c_2 exists and let $\Gamma_e = \Gamma[c_e]$ be an arbitrary step of odd height lying east of c_2 (if no such step exists, there is nothing to prove). Note that $\Gamma[c_2] \neq \Gamma_{bot}$ because of the existence of $\Gamma[c]$, and there is nothing to prove if $\Gamma[c_2] = \Gamma_{top}$. By Lemma 3, H may not end at c_2. This along with observation (c) above implies that $\overrightarrow{dd_n} \in H$ (otherwise H would end at c_2). Lemma 7 applied on $G_{ne}[d]$ tells us that either $h(d, \Gamma_e) = h(c_2, \Gamma_e)$ is even, or c_e is the end point of H. If $\Gamma_e \neq \Gamma_{top}$, then by Lemma 3 c_e cannot be the endpoint of H, so condition (2) holds. Assume now that $\Gamma_e = \Gamma_{top}$ and that the width of $G_{se}[c_2]$ is odd. In this case the rightmost edge in σ_d points south, and by Lemma 6 H cannot end in Γ_{top}. This implies that c_e is not the endpoint of H, so condition (2) holds.

Condition (3). Assume that c_2 exists and the width of $G_{se}[c_2]$ is even. Note that $\Gamma[c_2] \neq \Gamma_{bot}$ due to the existence of $\Gamma[c]$. We prove by contradiction that the height of $G_{se}[x]$ is also even. Assume to the contrary that the height of $G_{se}[x]$ is odd. In this case the lowest edge in σ_x points west, and by Lemma 6 the end point of H lies in Γ_{bot}. This along with observation (c) above implies that $\overrightarrow{dd_n} \in H$ (otherwise H would end at c_2). Because $\overrightarrow{dd_n} \in H$ and the width of $G_{se}[c_2]$ is even (by the case statement), the rightmost edge in σ_d points north (see Figure 15a). By Lemma 6 the end

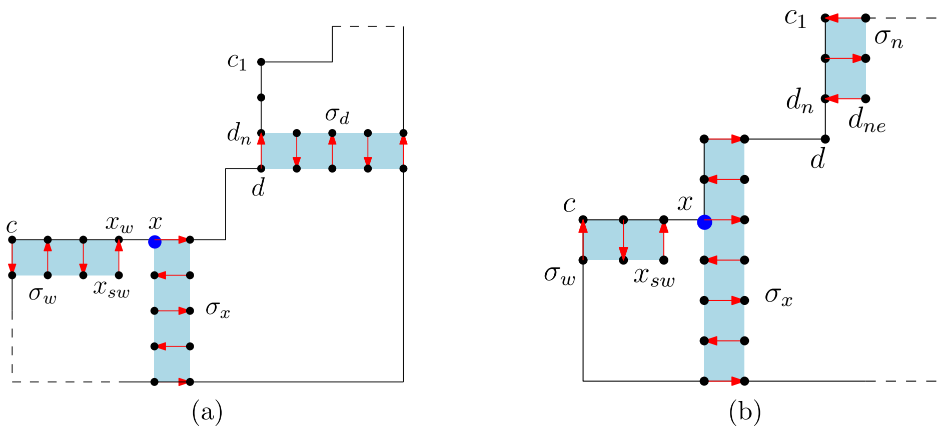


Figure 15: Theorem 12, only-if direction (a) $\overrightarrow{dd_n}$ exists (b) $\overrightarrow{dd_n}$ does not exist.

point of H lies in Γ_{top}, contradicting the fact (established above) that H ends in Γ_{bot}.

Condition (4). This proof that $|xc|$ is odd is identical to the proof for condition (4) in Theorem 11.

Condition (5). Assume that $|xc|$ is odd. By condition (4), $\Gamma[c] = \Gamma_{bot}$. Because $|xc|$ is odd, $\overrightarrow{c_s c} \in \sigma_w$, so H ends at c. If the height of $G_{se}[x]$ is even, then the lowest edge in σ_x points east, and by Lemma 6 H does not end in Γ_{bot}, a contradiction. It follows that $G_{se}[x]$ has odd height.

Condition (6). Assume that c_2 exists and $h(x, c_2) = 0$, meaning that x is the lower left corner of $\Gamma[c_2]$. Because the height of $\Gamma[c_2]$ is odd, the highest edge in σ_x is $\overrightarrow{c_{2e} c_2}$, so H ends at c_2. If the lowest edge in σ_x points west, then by Lemma 6 H ends in Γ_{bot}, a contradiction. So the lowest edge in σ_x points east, which by the definition of a zigzag sequence is possible only if the height of $G_{se}[x]$ is even. So condition (6) of the theorem holds. This completes the proof. □

10 Running Time Details from Section 5

We show that there is an $O(t)$ time algorithm for deciding whether a staircase grid graph G with t steps contains a squeeze-free Hamiltonian path that starts at a boundary vertex of ∂G. We assume G is represented as a sequence of pairs of numbers indicating the height and width of each step in order from left to right.

Theorems 10 through 12 list the conditions necessary and sufficient for the existence of a squeeze-free Hamiltonian path H that starts at a given vertex x on a horizontal staircase segment, and begins in the west, east or south direction. The cases with H beginning north from x, or with x on a vertical staircase segment, are symmetric. Therefore it suffices to show that we can determine in $O(t)$ time whether or not the conditions listed by Theorems 10 through 12 hold for at least one vertex x located on a horizontal staircase segment.

We begin by introducing two decision variables that will play a critical role in our decision procedure. For a fixed vertex $x \in \partial G$, define

$$v(x) = \begin{cases} 0 & \text{if } v(x, \Gamma_w) \text{ is even for each step } \Gamma_w \neq \Gamma_{bot} \\ & \text{of odd width lying west of } x \\ 1 & \text{if } v(x, \Gamma_w) \text{ is odd for each step } \Gamma_w \neq \Gamma_{bot} \\ & \text{of odd width lying west of } x \\ -1 & \text{otherwise} \end{cases}$$

Similarly, define

$$h(x) = \begin{cases} 0 & \text{if } h(x, \Gamma_e) \text{ is even for each step } \Gamma_e \neq \Gamma_{top} \\ & \text{of odd height lying east of } x \\ 1 & \text{if } h(x, \Gamma_e) \text{ is odd for each step } \Gamma_e \neq \Gamma_{top} \\ & \text{of odd height lying east of } x \\ -1 & \text{otherwise} \end{cases}$$

Observe that Theorems 10 through 12 are concerned with the parity of distances, not with the actual distances. We will keep track of these parities by taking all distances modulo 2 (with 0 representing even and 1 representing odd). In addition to the two variables defined above for all vertices x (including the special corner vertices c_1 from Theorem 11 and c_2 from Theorem 12), Theorems 10 through 12 employ the following decision variables: $h(x, \Gamma_{top})$; $v(x, \Gamma_{bot})$; parity of width and height of $G_{se}[x]$; $x \in \Gamma_{bot}$; parity of $|xc|$; existence and location of c_1 and c_2; height of $G_{se}[c_1]$; width of $G_{se}[c_2]$; and $\Gamma[c_2] = \Gamma_{top}$.

We begin by determining the values of these decision variables for the top left corner vertex of each step and then show that this is sufficient to determine if any vertex on a horizontal staircase segment satisfies the conditions of Theorems 10 through 12. Our method of computing these variables consists of two stages: a preprocessing stage, and an incremental update stage. The preprocessing stage initializes all variables corresponding to the top left corner of Γ_{bot}. In addition, it sets up some helper variables that will be used in the incremental update stage. In the incremental update stage, the top left corners of the steps are processed from left to right, and the variable values for the current corner are determined from the values of the previous corner in constant time.

Preprocessing

Let $\Gamma[x_1] = \Gamma_{bot}, \Gamma[x_2], \ldots, \Gamma[x_t] = \Gamma_{top}$ be the steps in order from left to right. Imagine a vertical line sweeping left-to-right across the steps, stopping at each corner vertex $x_1, x_2, \ldots, x_t$. Let o_1 be the top right corner of the first step of odd width encountered after Γ_{bot}. At each step $\Gamma[x_i]$ of odd width north of o_1, check the vertical distance from x_i to the previously visited step of odd width. If even, continue; if odd, let $p_1 = x_i$ and halt the sweeping process. Note that, for any vertex x at or above p_1, $v(x) = -1$; and for any vertex x above o_1 and strictly below p_1, $v(x)$ is either 0 or 1. For vertex o_1 and all vertices west of it, $v(x)$ is undefined because there is no step of odd width lying west of these vertices (except possibly Γ_{bot}).

Similarly, imagine a vertical line sweeping right-to-left across the steps. Let o_2 be the bottom left corner of the first step of odd height encountered after Γ_{top}. At each step $\Gamma[x_i]$ of odd height south of o_2, check the horizontal distance from x_i to the previously visited step of odd height. If even, continue; if odd, let $p_2 = x_i$ and halt the sweeping process. Note that, for any vertex x left of p_2, $h(x) = -1$; and for any vertex x at or to the right of p_2, $h(x)$ is either 0 or 1.

We create a list of all steps of odd height, to be used in determining the existence and position of c_2. We also determine the values for all decision variables corresponding to x_1. Note that every part of this preprocessing stage can be easily implemented in $O(t)$ time.

Incremental Update

Imagine a vertical line starting at x_2 and sweeping left-to-right across the steps, stopping at each top left corner vertex and initializing its decision variables. Let $width(\Gamma[x_i])$ and $height(\Gamma[x_i])$ be the width and height of step $\Gamma[x_i]$. At each corner x_i encountered by the sweep line, we initialize a selection of its decision variables as follows:

- $h(x_i)$: If o_2 does not exist, or if the sweep line has already passed o_2, this variable is undefined (since there are no steps of odd height east of x_i). If x_i is left of p_2, then $h(x_i) = -1$. If $x_i = p_2$, initialize $h(x_i) = 1$ (since the horizontal distance from p_2 to the closest step of odd height lying east of p_2 is odd, by the definition of p_2). If x_i is strictly right of p_2, update $h(x_i) = (h(x_{i-1}) + width(\Gamma[x_{i-1}])) \mod 2$.
- $v(x_i)$: If o_1 does not exist, or if the sweep line has not yet reached o_1, this variable is undefined (since there are no steps of odd width west of x). If the sweep line is at o_1, initialize $v(x_i) = height(x_i) \mod 2$. If the sweep line has passed p_i, $v(x_i) = -1$; otherwise, update $v(x_i) = (v(x_{i-1}) + height(\Gamma[x_i])) \mod 2$.
- $h(x_i, \Gamma_{top})$ and width of $G_{se}[x_i]$: set to the value of the variable for x_{i-1} incremented by $width(x_{i-1})$ (modulo 2).
- $v(x_i, \Gamma_{bot})$ and height of $G_{se}[x_i]$: set to the value of the variable for for x_{i-1} incremented by $height(x_i)$ (modulo 2).

Next we restart the sweeping process to update the remaining decision variables in O(1) time (per step). Note that testing if $\Gamma[x_i] = \Gamma_{bot}$ and $\Gamma[c_2] = \Gamma_{top}$ can be easily determined in constant time, and $|x_i c|$ is zero for all top left corner step vertices. The only decision variables left concern the existence and location of c_1 and c_2, which are initialized for each x_i in $O(1)$ time as follows:

- existence of c_1: true if o_1 exists and the sweep line has passed it, false otherwise. If x_i is the corner vertex of a step of odd width, update a temporary copy $c_1 = x_i$, to become permanent once a new corner vertex is encountered.
- existence of c_2: true if o_2 exists and the sweep line has not reached it yet, false otherwise. We maintain a pointer to the step $\Gamma[c_2]$ (and the associated width of $G_{se}[c_2]$) in the list of steps of odd height. If x_i coincides with c_2, advance the pointer.

Having determined c_1 and c_2, we can access in constant time the values $v(c_1)$, $v(c_1, \Gamma_{bot})$, height of $G_{se}[c_1]$, $h(c_2)$, $h(\Gamma_{top})$, and the width of $G_{se}[c_2]$, computed in the previous sweep stage.

Running Time

The preprocessing stage and incremental update stage for computing the values of the decision variables for each step's top left corner run in $O(t)$ time. Observe that for each decision variable, its value is either the same for all the vertices on a stair's horizontal top segment or its value alternates between 0 and 1 (as the distance of the vertex from the top left corner of the step alternates between even and odd). Therefore, for any step $\Gamma[c]$, the values of the decision variables for c are the same as the values for c_{ee}, and the values for c_e are the same as the values for c_{eee}, and so on... This means it is only necessary to check the conditions listed by Theorem 10 for vertices c_e and c_{ee}, because the variable values for all the other vertices on the step will be the same as for one of these two vertices. For Theorems 11 and 12, it is only necessary to check the conditions for vertices c and c_e. After computing the values of the variables for c using the algorithm described, the variable values for c_e, c_{ee} can easily be determined in constant time because, depending on the variable, they are either the same or the opposite value as that for c.

Thus, once the variable values are calculated for each step's top left corner, we can determine in constant time per step if there is any vertex on its top horizontal segment that satisfies the conditions of Theorems 10- 12. This gives us the result in Theorem 13 from Section 5.