Tradeoffs between Bends and Displacement in Anchored Graph Drawing

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Abstract

Many graph drawing applications entail geographical constraints on positions of vertices; these constraints can be at odds with aesthetic requirements such as the use of straight-line edges or the number of crossings. Without positional constraints on vertices, of course, every planar graph can be drawn crossing-free with straight-line edges. On the other hand, inflexible and precise specification of all vertex positions essentially leaves no room for presenting the graph in an aesthetically pleasing drawing. However, small deviations from precise vertex positions can often be tolerated, and so a natural middle ground is to impose *soft positional constraints* on vertices and then optimize for an appropriate aesthetic criterion.

We explore one such trade-off: the amount of vertex position displacement vs. the number of bends in planar polyline drawings. In particular, let $G = (V, E)$ be a planar graph, where each vertex v has a specified (target) position $\alpha(v)$. We wish to draw G so that no vertex is placed at distance more than δ from its target position and no edge has more than b bends. Given a bound on b, what is the smallest value of δ achievable for all n-vertex planar graphs? Our main result establishes that $\delta = \Theta(n)$ is both necessary and sufficient if b is constant. We also derive trade-offs between δ and b.

1 Introduction

Visual representations of graphs face multiple, often conflicting, constraints. This paper explores one such trade-off: the tension between aesthetic aspects of a graph drawing and its informational distortion. Specifically, we have a planar graph $G = (V, E)$ on $n = |V|$ vertices, where each vertex v has a specified (target) position $\alpha(v)$ in the plane. Such positional constraints naturally arise in many geo-spatial datasets, such as positions of cities or municipalities in country maps. Positional constraints also arise when the graph is visualized in a larger context: for instance, if the graph is to be overlaid on another graph with a common or overlapping set of entities, then a close correspondence of vertices is highly desirable; the same holds if both graphs are shown next to each other or one after the other. In all these scenarios, placing vertices far away from their intended position can create loss of information and readability since it distorts the user's knowledge of positions (the mental map).

Unfortunately, such positional constraints on vertices can be at odds with other aspects of the drawing, such as aesthetics and readability. For instance, every planar graph can be drawn with straight-line edges and no edge crossings [4], but doing so requires the freedom to move vertices in the drawing space. On the other hand, fixing each vertex's position precisely does not leave much room for an informative drawing: indeed, the vertex positions fix the straight-line drawing and also the number of edge crossings. A natural middle ground, therefore, is to treat the vertex position constraints as *soft constraints*, allowing the flexibility to place the vertices close to their ideal position while improving the quality of the drawing.

Our paper is an exploration of one such trade-off. We ask how much better can the drawing be made if each vertex v is allowed to be displaced by some distance δ from its target position $\alpha(v)$. All the edges must be drawn as polylines with no crossings, and no polyline has more than b *bends*, which we call the *curve complexity* of the drawing. (The curve complexity of a straight-line drawing is $b = 0$.) Our trade-off explores how much benefit in terms of the curve complexity one can expect by increasing the displacement as a function $\delta(n)$; more precisely, given a maximum displacement δ, what is the smallest curve complexity $b(\delta, n)$ achievable for all n-vertex planar graphs? Similarly, for a given curve complexity b, we want to know the smallest displacement $\delta(b, n)$ that is sufficient for all n-vertex graphs. We call our problem the *anchored graph drawing* problem because each vertex has an ideal (anchor) position.

Previous Work. Our research touches two important topics in graph drawing: positional constraints on vertices and bend-minimization in polyline drawings. If each vertex has a disk-shaped region within which it must be placed, then it is NP-hard to decide if a straight-line planar drawing exists, as Godau showed [5]. In a followup work, Angelini et al. showed that the problem remains NP-hard even if the disk regions of all vertices have the same radius [1]. Extending these hardness results, Löffler [9] showed that it is NP-hard to decide if a straight-line embedding exists for regions that are vertical line segments, even if the graph is only a cycle.

When the displacement shrinks to $\delta = 0$ vertex move-

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ments are disallowed; this yields the classic point-set embeddability problem with fixed vertex-point mapping. In contrast to the version without given mapping, the straight-line problem is trivial since there is only one straight-line drawing which either is or is not planar. However, there are several results for polyline edges.

Pach and Wenger [10] showed that, in polynomial time, every planar graph can be embedded with fixed vertex positions and $O(n)$ bends per edge. Furthermore, they proved that if the points are in convex position, for every planar graph, the probability is high that a linear number of edges will need a linear number of bends. For non-convex position, Badent et al. [2] constructed a family of instances in which a linear number of edges must have a linear number of bends. In a recent arxiv submission, Gordon [6] shows that for every set of vertex positions, a planar graph that is sampled uniformly at random (with fixed vertex-point mapping) will require $\Omega(n^2)$ bends in total with high probability.

Point-set embeddability has also been considered without prescribed vertex-point mapping. In this setting, a set of n points is prescribed and for each vertex one of the input points must be chosen as the vertex position. Gritzmann et al. [7] introduced this problem class and showed that a planar straight-line embedding can be constructed for every outerplanar graph. However, for general planar graphs, it is NP-hard to decide whether a planar straight-line embedding exists, as Cabello proved [3]. Kaufmann and Wiese [8] showed that for every planar graph a vertex-point mapping can be found such that the graph can be embedded with only two bends per edge (one bend for four-connected graphs).

Our Results. Our work differs from these earlier lines of research in that we explore the trade-off between positional displacement δ and maximum number b of bends per edge (the *curve complexity*) for which a feasible planar drawing exists. More specifically, given a curve complexity b, we ask for the smallest displacement δ such that a feasible anchored drawing exists. (b and δ may be constants or functions of n.) When $\delta = 0$, we get the classic point-set embeddability problem. In our problem, however, vertices can be displaced within the distance bound δ, and we wish to explore the effect of δ on the curve complexity b that is necessary for a plane drawing. Since we must relate the value of δ to the area of the drawing and the distance between vertices, we usually assume that the target vertex positions are points of the integer $n \times n$ grid, so that the smallest distance between vertices is 1. However, we do not demand that in the final drawing vertices and bends have integer coordinates.

We will see that even with a positive (but small) value of δ, there are graphs and target positions for which a linear number of edges require a linear number of bends in every feasible drawing. If we allow displacement $\delta = O(n)$, then an easy construction can achieve $b = 0$, namely, a straight-line embedding, for any n-vertex planar graph. With some more effort and care, we can show that curve complexity $b = 2$ is always possible for $\delta > (n-1)/2$.

Our main result is to show that, surprisingly, this linear displacement is necessary for any constant number of bends. More specifically, we show that for any constant b, there are planar graphs that require a minimum vertex displacement of $\Omega(n)$ to realize a polyline drawing with at most b bends. In fact, if the vertex displacement is $o(n)$, then at least $\Omega(n)$ edges require more than b bends. We also show that for curve complexity $b = \Theta(\sqrt[3]{n})$, a displacement of $\Omega(\sqrt[3]{n})$ is necessary, and that for any constant displacement, there are instances that force a curve complexity of $\Omega(\sqrt{n})$.

2 Preliminaries

We call our problem the Anchored Graph Drawing Problem (AGD), following Angelini et al. [1]. In addition to planar input graph and target vertex positions, the problem takes two parameters: the maximum displacement δ of vertices from their target position and the curve complexity b. Since we are interested in the relation between δ and b, we call the problem δ-b-AGD.

Problem (δ-b-AGD) *Given a planar graph $G = (V, E)$ with $n = |V|$, a function $\alpha\colon V \to \mathbb{N} \times \mathbb{N}$ that assigns to each vertex v a position $\alpha(v)$ on the $n \times n$ grid, a number $\delta \in \mathbb{R}^+$, and a number $b \in \mathbb{N}$ find a planar polyline drawing $\mathcal{E}$ of G such that every vertex v is placed within distance δ of $\alpha(v)$ and no edge has more than b bends.*

We call a feasible embedding for such an instance a δ-b-AGD embedding. We will sometimes speak of *moving* a vertex v to mean that the vertex is placed within distance δ of its target position $\alpha(v)$. Similarly, a δ*-movement* of the vertices allows to place each vertex at a position up to δ from its target position.

Depending on instance and parameters, it is not clear whether a δ-b-AGD embedding exists. Hence, the question is how the parameters relate to each other and to n. For given b and n, we would like to know how big δ must be so that every instance of n vertices has a δ-b-AGD embedding. To this end, we define two values.

Definition 1 *Let $G = (V, E)$ be a planar graph with $n = |V|$ and let $\alpha\colon V \to \mathbb{N} \times \mathbb{N}$ define target positions for the vertices on the $n \times n$ grid. Let $b \geq 0$ be the curve complexity. We define $\delta(b, G, \alpha)$ to be the minimum value δ such that a δ-b-AGD embedding of G exists.*

Now, we consider the relation between b, n, and δ.

Definition 2 ($\delta(b, n)$) *Let $b, n \geq 0$ be integer values. We define $\delta(b, n)$ to be the maximum value $\delta(b, G, \alpha)$ over all instances of a planar graph*

G with n vertices and target positions α, i.e., $\delta(b,n) = \max\{\delta(b,G,\alpha) \mid G=(V,E)$ *planar*, $|V|=n$, $\alpha\colon V \to \mathbb{N}\times\mathbb{N}\}$.

In the following sections, we will find upper and lower bounds for $\delta(b,n)$. Since it turns out that for every constant b the lower bound for $\delta(b,n)$ is linear in n, it makes sense to also consider values for b that depend on the size of the graph; then, b can be a function of n.

Analogously, we can define values corresponding to the minimum number of bends for which a δ-b-AGD embedding with given δ exists.

Definition 3 *Let $G=(V,E)$ be a planar graph on n vertices and let a function $\alpha\colon V\to\mathbb{N}\times\mathbb{N}$ define target positions for the vertices of G on the $n\times n$ grid. Let $\delta\geq 0$ be the maximum vertex displacement. We define $b(\delta,G,\alpha)$ to be the minimum curve complexity $b\geq 0$ such that a δ-b-AGD embedding of G exists.*

Definition 4 ($b(\delta,n)$) *Let $n\geq 0$ be an integer value and let $\delta\geq 0$. We define $b(\delta,n)$ to be the maximum value $b(\delta,G,\alpha)$ over all instances of a planar graph G with n vertices and target positions α, i.e., $b(\delta,n) = \max\{b(\delta,G,\alpha) \mid G=(V,E)$ planar, $|V|=n$, $\alpha\colon V\to\mathbb{N}\times\mathbb{N}\}$.*

3 Upper Bounds

We recall that even without displacement of the vertices (i.e. with fixed vertex positions), every planar graph can be embedded with curve complexity $O(n)$ using the algorithm of Pach and Wenger [10]. Thus, there is always a δ-$O(n)$-AGD embedding, no matter how small δ is. Our lower bound result will later (cf. Theorem 8) establish that a linear curve complexity is necessary even if we allow positive displacement of vertices. We begin with our upper bounds for $\delta(b,n)$.

By choosing $\delta=\sqrt{2}(n-1)$, any vertex can be placed freely in the area spanned by the $n\times n$ grid, thus effectively removing the restriction of the δ-movement and allowing to use any algorithm for creating a planar straight-line embedding. Since the final vertex positions after the movement do not have to lie on the grid, any value $\delta>\sqrt{2}(n-1)/2=(n-1)/\sqrt{2}$ is sufficient; such a value allows all vertices to be moved to and within a small area around the center of the $n\times n$ grid.

Observation 1 *For any $\varepsilon>0$, $\delta(0,n)\leq(n-1)/\sqrt{2}+\varepsilon$, that is, for $\delta=(n-1)/\sqrt{2}+\varepsilon$, there is a δ-0-AGD embedding for every planar graph whose target positions lie on the $n\times n$ grid.*

If we allow two bends per edge, a smaller bound on δ can be shown, using the following result of Kaufmann and Wiese [8]. (This is not explicitly stated in their paper, but follows from their point-set embeddability construction without prescribed vertex-point mapping.)

Lemma 1 ([8]) *For every planar graph $G=(V,E)$ there is an ordering $V=\{v_1,\dots,v_n\}$ of the vertices, such that for any assignment of coordinates that follows the left-to-right order $x(v_1)<x(v_2)<\ldots<x(v_n)$ a planar 2-bend embedding can be found.*

Using this lemma, we can prove the following result.

Theorem 2 *For any $\varepsilon>0$, $\delta(2,n)\leq(n-1)/2+\varepsilon$, that is, for every planar graph $G=(V,E)$ with n vertices whose target positions $\alpha\colon V\to\mathbb{N}\times\mathbb{N}$ lie on the $n\times n$ grid there is a δ-2-AGD embedding with $\delta=(n-1)/2+\varepsilon$.*

Proof. Let $V=\{v_1,\dots,v_n\}$ be the order of vertices achieved by Lemma 1. If we can find a δ-movement that orders $v_1,\dots,v_n$ from left to right, then a 2-bend embedding follows. We achieve the ordering as follows.

By moving all vertices horizontally by at most $(n-1)/2$ we can put them on a vertical line through the middle of the $n\times n$ grid. With the remaining movement of ε, we create the correct left-to-right order. With these vertex positions, the 2-bend embedding can be created. □

Since b bends are also feasible if higher curve complexity would be allowed, we also get bounds for other complexities. Summarizing, we get the following result.

Theorem 3 *For any $\varepsilon>0$, it holds $\delta(1,n)\leq\delta(0,n)\leq(n-1)/\sqrt{2}+\varepsilon$ and $\delta(b,n)\leq(n-1)/2+\varepsilon$ for $b\geq 2$.*

Our upper bound for δ does not improve with larger values of b. Finding a construction for general b is an interesting open problem.

We now present the main result of our paper, which is a family of lower bounds for $\delta(b,n)$. Somewhat surprisingly it turns out that the $O(n)$ vertex displacement, so easily achieved by our upper bound above, cannot be improved, namely, $\delta(b,n)=\Omega(n)$ for any constant value of b.

4 Lower bounds

We will construct negative examples that show that even for relatively large values of δ, depending on b, no feasible δ-b-AGD embeddings exist. More precisely, for every number b (from constant to $\Theta(n)$), we find a family of planar graphs and point sets with according δ, such that in every feasible embedding with a δ-movement of the vertices, a *linear number* of edges will have more than b bends. Our proof is constructive and draws inspiration from the bad instances of point-set embeddability used by Badent et al. [2].

Theorem 4 *Let $b\geq 0$. Then, for $n\geq 4\sqrt{2}(4b+5)/\pi^2+1$ it holds that $\delta(b,n)\geq(n-1)\pi^2/(16(4b+5)^2)$, that is, $\delta(b,n)=\Omega(n/b^2)$. More precisely, for any such n there is an example in which a linear number of edges must have more than b bends if the vertices are moved by at most $(n-1)\pi^2/(16(4b+5)^2)$.*

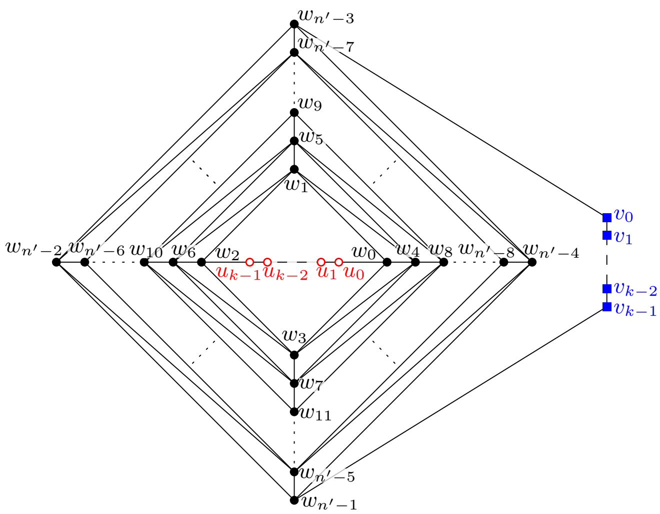

Figure 1: Schematic visualization of the graph used in our constructions for lower bounds. Except for the choice of the outer face, this is the only possible embedding.

Proof. The main idea is to have many 4-cycles such that, in every planar embedding, every 4-cycle has to separate two sets of *red* and *blue vertices* of equal size into vertices inside and outside of the cycle. By carefully placing the target positions of these red and blue vertices and choosing the right value δ, we achieve that even after a δ-movement of the vertices every 4-cycle separating the red and blue vertices must be realized as a complex polygon, having at least one edge with 1/4 of the necessary bends.

Our graph is constructed as follows; see Fig. 1. Let n' be a multiple of 4, and let $k \geq 1$. The graph consists of a set of n' black vertices $V_0 = \{w_0, w_1, \dots, w_{n'-1}\}$, a set of k *red* vertices $V_1 = \{u_0, u_1, \dots, u_{k-1}\}$, and a set of k *blue* vertices $V_2 = v_0, v_1, \dots, v_{k-1}\}$. The set of edges consists of five subsets, i.e., $E = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$, which are defined as follows.

$$\begin{aligned}
E_0 &= \{(w_0, u_0), (u_0, u_1), \dots, (u_{k-2}, u_{k-1}), (u_{k-1}, w_2)\} \\
E_1 &= \{(w_{n'-3}, v_0), (v_0, v_1), \dots, (v_{k-1}, w_{n'-1})\} \\
E_2 &= \{(w_i, w_{i+1}), (w_{i+1}, w_{i+2}), (w_{i+2}, w_{i+3}), (w_{i+3}, w_i) \\
&\qquad \mid 0 \leq i < n', i \bmod 4 = 0\} \\
E_3 &= \{(w_{i+4}, w_{i+1}), (w_{i+1}, w_{i+6}), (w_{i+6}, w_{i+3}), (w_{i+3}, w_{i+4}) \\
&\qquad \mid 0 \leq i < n' - 4, i \bmod 4 = 0\} \\
E_4 &= \{(w_i, w_{i+4}) \mid 0 \leq i < n' - 4\}
\end{aligned}$$

The edges in E_0 form a path from w_0 to w_2 containing all red vertices and the edges in E_1 form a path from $w_{n'-3}$ to $w_{n'-1}$ containing all blue vertices. The edges of E_2 and E_3 form $n'/4$ and $n'/4 - 1$ independent 4-cycles, respectively. If we replace the paths $(w_0, u_0, \dots, u_{k-1}, w_2)$ and $(w_{n'-3}, v_0, \dots, v_{k-1}, w_{n'-1})$ both by a single edge, the graph is triangulated; hence, up to the choice of the outer face, there is only one combinatorial embedding, the one in Fig. 1. Therefore, each of the $2n'/4 - 1$ edge-disjoint 4-cycles defined by E_2 and E_3 must separate the red vertices from the blue vertices outside in any planar embedding.

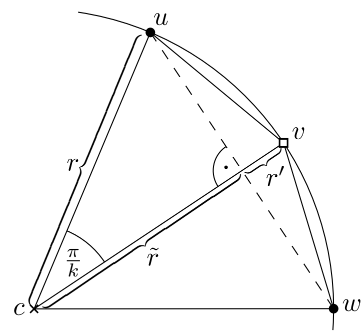

Figure 2: Angles and distances in the regular $2k$-gon.

In our target positions, the red and blue vertices will form a *bi-colored sequence* $(u_0, v_0, u_1, v_1, \dots, u_{k-1}, v_{k-1})$. Since every pair of consecutive vertices in the sequence has different colors, each 4-cycle must cross the straight-line segment connecting the vertices in order to separate them. Badent et al. [2] arranged the bi-colored sequence as consecutive points on a straight-line. However, if $\delta' > 0$, a δ'-movement that moves all red vertices down and all blue vertices up, easily allows the points to be separated by a single straight-line segment. Hence, we need a different construction in order to ensure that even after a δ-movement the vertices are hard to separate.

We do so by putting the points of the bi-colored sequence at the corners of a regular $2k$-gon with circumradius $r = (n-1)/2$ centered in the center c of the $n \times n$ grid. We want that after a δ'-movement of the vertices the $2k$-gon must still be convex, no matter how the vertices are moved. To this end, consider the relative positions of three consecutive vertices u, v, and w in the bi-colored sequence; see Fig. 2. As long as v stays on the same side of the line $\overline{uw}$, the angle at v remains convex. Since we can move both this line (by moving u and w) and v, we therefore require that $\delta' \leq r'/2$. We have $\cos(\pi/k) = \tilde{r}/r$. Hence,

$$\delta' \leq \frac{r'}{2} = \frac{r - \tilde{r}}{2} = \frac{r - r\cos(\pi/k)}{2} = \frac{r \cdot (1 - \cos(\pi/k))}{2}.$$

The Taylor series definition of the cosine function yields $\cos(\pi/k) \geq 1 - (\pi/k)^2/2 = 1 - \pi^2/(2k^2)$. Hence, $\delta' \leq (r\pi^2)/(4k^2)$. For any such δ' any δ'-movement of the vertices of the bi-colored sequence results in a convex polygon defined by the sequence in the input order.

We now modify the construction for the bi-colored sequence so that all target positions lie on points of the $n \times n$ grid. We assume that n is sufficiently large so that we can feasibly set $r\pi^2/4k^2 \geq \delta' \geq \sqrt{2}/2$. Let $\delta = \delta'/2$. We move each vertex of the $2k$-gon to the nearest grid point, which is at most $\sqrt{2}/2$ away. In any feasible solution, the distance of a vertex to its position on the

curve complexity b	$\Theta(1)$	$\Theta(\sqrt[3]{n})$	$\Theta(\sqrt{n})$
$\delta(b,n)$	$\Omega(n)$	$\Omega(\sqrt[3]{n})$	$\Omega(1)$

Table 1: Lower bounds for the displacement δ depending on the curve complexity b from Theorem 4.

$2k$-gon—resulting both from moving the vertex and from placing it on a grid point—is at most $\delta + \sqrt{2}/2 \leq \delta'$. Hence, the bi-colored sequence forms a convex polygon.

After placing the target positions for the bi-colored sequence, we place the target positions of the remaining vertices on unused points of the $n \times n$ grid.

The intersection of a straight line with a convex polygon is a straight-line segment, and, hence, each edge segment crosses the polygon's boundary at most twice.

Property A polyline that crosses the boundary of a convex polygon b times has at least $\lceil b/2 \rceil - 1$ bends.

On the other hand, each 4-cycle must separate the two sets of the bi-colored sequence and especially every pair of consecutive vertices; hence, every 4-cycle must cross each of the $2k$ edges of the corresponding convex polygon. Therefore, at least one of the edges of such a 4-cycle must have at least $\lceil 2k/4 \rceil = \lceil k/2 \rceil$ crossings with the boundary of the convex polygon. This edge must, hence, have at least $\lceil (\lceil k/2 \rceil)/2 \rceil - 1 = \lceil k/4 \rceil - 1$ bends.

Recall that $\delta \leq r\pi^2/(8k^2) = (n-1)\pi^2/(16k^2)$. Since we want to create an instance where each 4-cycle has an edge with at least $b+1$ bends, we must have $k \geq 4b+5$; we choose $k = 4b+5$. Hence, we can set $\delta = (n-1)\pi^2/(16 \cdot (4b+5)^2) = \Theta(n/b^2)$. Recall that we required that $r\pi^2/4k^2 \geq \sqrt{2}/2$. This is the case if $n \geq 4\sqrt{2}(4b+5)^2/\pi^2 + 1$. Furthermore, since there is a total of $2k = 8b+10$ red and blue vertices, there is a linear number $n' = n - 2k$ of remaining vertices forming the 4-cycles. Hence, in any planar drawing with just a δ-movement of the vertices, there will be a linear number of edges with more than b bends. $\square$

The general form of the theorem allows to choose the curve complexity b, as long as $n \geq 4\sqrt{2}(4b+5)^2/\pi^2 + 1$. This yields some interesting bounds; see also Table 1.

We first consider constant curve complexity. It is not surprising, that there are examples in which no constant curve complexity is sufficient. However, this is even the case with $\delta = \Theta(n)$, i.e., a linear size of δ may still be not enough freedom for constant curve complexity.

Corollary 5 *For every constant number $b \geq 0$ of bends and every number $n \geq 4\sqrt{2}(4b+5)^2/\pi^2 + 1$ it holds that $\delta(b,n) = \Omega(n)$. Furthermore, for every such n, there is an instance with $\delta = \Theta(n)$ such that in every feasible embedding $\Theta(n)$ edges must have more than b bends.*

Our construction also works for a curve complexity of $b = \Theta(\sqrt{n})$, and yields a constant δ-value.

Corollary 6 *Let $b = \Theta(\sqrt{n})$. For $n \geq 4\sqrt{2}(4b(n)+5)^2/\pi^2 + 1$ it holds that $\delta(b,n) = \Omega(1)$.*

Finally, both δ and b can be of $\Theta(\sqrt[3]{n})$; especially, both values are sublinear but not constant.

Corollary 7 *Let $b = \Theta(\sqrt[3]{n})$. For $n \geq 4\sqrt{2}(4b(n)+5)^2/\pi^2 + 1$ it holds that $\delta(b,n) = \Omega(\sqrt[3]{n})$.*

Note that Theorem 4 does not yield a bound for linear b. However, this restriction stems only from requiring that the target positions must lie on the grid. If we drop this requirement, we can place the bi-colored sequence on the corners of the regular $2k$-gon and get an example with a small—but positive—δ, for which a linear number of edges needs a linear number of bends. Note that, although the target positions do not lie on grid points, we still have the property that between every pair of vertices there is a larger distance, i.e., points do not come too close; in this case, the distance is at least constant.

Corollary 8 *Let b be a function linear in n. For every n with $n \geq 4b(n) + 9$, there is a planar graph with target positions (not lying on the $n \times n$ grid) and a value $\delta > 0$ such that every AGD embedding will have an edge with at least $b(n)$ bends.*

If $n - 4b = \Theta(n)$, these instances will even have a linear number of edges with a linear number of bends.

5 Bounds for $b(\delta, n)$

We now assume that δ is prescribed and derive bounds for the minimum $b(\delta,n)$ that is sufficient for all instances on n vertices. We reuse the constructions for $\delta(b,n)$. However, we must be careful with the modified analysis.

Upper Bounds. The constructions in Section 3 can be used directly for obtaining upper bounds on $b(\delta,n)$.

Theorem 9 *For $\delta > (n-1)/2$ it holds that $b(\delta,n) \leq 2$ and for $\delta > (n-1)/\sqrt{2}$ it holds that $b(\delta,n) = 0$.*

Lower Bounds. Using the examples of Section 4, we can also derive lower bounds on $b(\delta,n)$.

Theorem 10 *Let $\delta \geq \sqrt{2}/4$. Then, for every $n \geq 1$ it holds that $b(\delta,n) \geq \sqrt{(n-1)/\delta} \cdot \pi/16 - 1$, that is, $b(\delta,n) = \Omega(\sqrt{n/\delta})$.*

More precisely, there is always an example in which a linear number of edges must have at least $\sqrt{(n-1)/\delta} \cdot \pi/16 - 1$ bends if the vertices are moved by at most δ.

Proof. We can use the same construction as in Theorem 4. However, now the displacement δ is prescribed and we want to maximize k such that the resulting $2k$-gon will stay convex after any δ-movement of the vertices. We require $\delta \geq \sqrt{2}/4$ since otherwise moving the vertices

maximum displacement δ	$\Theta(1)$	$\Theta(\sqrt[3]{n})$	$\Theta(n)$
$b(\delta, n)$	$\Omega(\sqrt{n})$	$\Omega(\sqrt[3]{n})$	$\Omega(1)$

Table 2: Lower bounds for the curve complexity b depending on the displacement δ from Theorem 10.

from the corners of the regular $2k$-gon to the closest grid points can have more influence than the δ-movement.

Recall that we required $\delta \leq (n-1)\pi^2/(16k^2)$. Hence, we can set $k = \lceil \sqrt{(n-1)/\delta} \cdot \pi/4 \rceil$. Since every 4-cycle must have an edge with at least $\lceil k/4 \rceil - 1$ bends, we know that every feasible planar embedding with only a δ-movement of the vertices must have curve complexity at least $\lceil \sqrt{(n-1)/\delta} \cdot \pi/16 \rceil - 1$. Therefore, $b(\delta, n) \geq \lceil \sqrt{(n-1)/\delta} \cdot \pi/16 \rceil - 1$.

Since in the example we have $2k = O(\sqrt{n})$ vertices in the bi-colored sequence, there will still be a linear number of 4-cycles and, hence, a linear number of edges that need at least $\sqrt{(n-1)/\delta} \cdot \pi/16 - 1$ bends. $\square$

As a consequence, for every constant δ we get $b(\delta, n) = \Omega(\sqrt{n})$. Especially, no constant curve complexity can be guaranteed with a constant displacement.

Corollary 11 *For every constant displacement $\delta \geq 0$ and $n \geq 1$ it holds that $b(\delta, n) = \Omega(\sqrt{n})$.*

Furthermore, for every n, there is an instance with $b = \Theta(\sqrt{n})$ such that in every feasible δ-b-AGD embedding $\Theta(n)$ edges must have at least b bends.

Again, we can also make δ depend on n. We get essentially the same relation between δ and $b(\delta, n)$ as we did for b and $\delta(b, n)$; see Table 2.

Corollary 12 *For $\delta = \Theta(n)$, $b(\delta, n) = \Omega(1)$.*

Again, there are examples in which both δ and b are of $\Theta(\sqrt[3]{n})$, that is, both are sublinear but not constant.

Corollary 13 *For $\delta = \Theta(\sqrt[3]{n})$, $b(\delta, n) = \Omega(\sqrt[3]{n})$.*

6 Conclusion and Open Problems

We explored the interplay between flexibility in moving vertices away from their target position with the number of bends in planar drawings. We proved upper and lower bounds for the value $\delta(b, n)$ that describes the displacement that has to be allowed in order to be able to draw all planar instances with only b bends per edge. Most importantly, we have seen that for every constant curve complexity b, $\delta(b, n)$ is still linear. Furthermore, even $\Theta(\sqrt[3]{n})$ curve complexity is not achievable with constant displacement, but requires $\Omega(\sqrt[3]{n})$ displacement. On the other hand, we have also shown that any constant maximum displacement δ still requires $b(\delta, n) = \Omega(\sqrt{n})$.

There are still several interesting open questions. For instance, for higher constant values of b, the gap in terms of the constants for the upper and lower bounds is quite large. Is there an algorithm that finds a drawing with constant curve complexity b and relatively small linear displacement? Furthermore, we know that for curve complexity $\Theta(\sqrt{n})$ a constant displacement is necessary. Is such a displacement also sufficient, i.e., is there a constant δ such that we can draw every planar graph with displacement just δ and curve complexity $O(\sqrt{n})$?

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