Conflict-free Covering*

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Abstract

Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of color classes, where each color class C_i consists of a set of points. In this paper, we address a family of covering problems, in which one is allowed to cover at most one point from each color class. We prove that the problems in this family are NP-complete (or NP-hard) and offer several constant-factor approximation algorithms.

1 Introduction

Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of color classes, where each color class C_i consists of a set of points. In this paper we address several closely related covering problems, in which one is allowed to cover at most one point from each color class. Before defining the problems, let us introduce some terminology. Let the set P of point color classes be on a line. We call an interval on the x-axis that contains at most one point from each color class a *conflict-free* interval (or CF-interval for short).

In this paper we consider the following problems.

Covering color classes with CF-intervals: Given a set P of point color classes on a line where each color class consists of a pair, find a minimum-cardinality set $\mathcal{I}$ of CF-intervals, such that at least one point from each color class is covered by an interval in $\mathcal{I}$.

Covering color classes with arbitrary unit squares: Given a set P of point color classes in the Euclidean plane where each color class consists of a vertically or horizontally unit separated pair of points, find a minimum-cardinality set $\mathcal{S}$ of unit squares (assuming a feasible solution exists), such that exactly one point from each color class is covered by a square in $\mathcal{S}$.

Covering color classes with a convex polygon: Given a set P of point color classes in the Euclidean plane where each color class consists of either a pair or a triple of points, decide whether or not there exists a convex polygon Q such that Q contains exactly one point from each color class. We also consider the related problem in which each color class consists of a pair of points and the goal is to maximize the number of color classes covered by a convex polygon Q, with Q containing exactly one point from each color class.

1.1 Related work

As far as we know, the first to consider a "multiple-choice" problem of this kind were Gabow et al. [7], who studied the following problem. Given a directed acyclic graph with two distinguished vertices s and t and a set of k pairs of vertices, determine whether there exists a path from s to t that uses at most one vertex from each of the given pairs. They showed that the problem is NP-complete. A sample of additional graph problems of this kind can be found in [2, 8, 13]. The first to consider a problem of this kind in a geometric setting were Arkin and Hassin [3], who studied the following problem. Given a set V and a collection of subsets of V, find a cover of minimum diameter, where a cover is a subset of V containing at least one representative from each subset. They also considered the multiple-choice dispersion problem, which asks to maximize the minimum distance between any pair of elements in the cover. They proved that both problems are NP-hard and gave $O(1)$-approximation algorithms. Recently, Arkin et al. [1] considered the following problem. Given a set S of n pairs of points in the plane, color the points in each pair by red and blue, so as to optimize the radii of the minimum enclosing disk of the red points and the minimum enclosing disk of the blue points. In particular, they consider the problems of minimizing the maximum and minimizing the sum of the two radii. In another recent paper, Consuegra and Narasimhan [4] consider several problems of this kind.

1.2 Our results

In Section 2 we consider the problem dealing with covering color classes, each consisting of a pair of points, with a minimum-cardinality set of CF-intervals. We prove that it is NP-hard by first proving that the following problem (covering color classes with a *given* set of CF-intervals) is NP-hard. Given a set P of point color classes and a set $\mathcal{I}$ of CF-intervals, find a minimum-

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cardinality set $\mathcal{I}' \subseteq \mathcal{I}$ (if it exists), such that, at least one point from each color class is covered by an interval in $\mathcal{I}'$. The latter proof is by a reduction from minimum vertex cover. The former proof also requires the following auxiliary result, which we state as an independent theorem. More precisely, we prove that minimum vertex cover remains NP-hard even when we restrict the underlying set of graphs to graphs in which each vertex is of degree at least $|V|/2$, where V is the set of vertices of the graph. We present a 4-approximation algorithm for this problem. We also present a 2-approximation algorithm for covering with arbitrary CF-intervals.

In Section 3 we consider the case where P is a set of point color classes in the Euclidean plane.

Suppose each color class consists of a pair and each pair of points from the same color class is unit distance apart, either vertically or horizontally separated. We show that finding a minimum-cardinality set $\mathcal{S}$ of axis parallel unit squares (assuming a feasible solution exists), such that exactly one point from each color class is covered by a square in $\mathcal{S}$ is NP-hard. We then present a 6-approximation algorithm.

We then consider the case that each color class consists of either a pair or triple of points. We show that deciding if there exists a convex polygon Q such that Q contains exactly one point from each color class is NP-complete. If each color class consists of a pair of points, we show that maximizing the number of color classes covered by Q is NP-hard. Finally, we consider the case that each color class consists of an arbitrary amount of points and all points from the same color class are vertically collinear. We (optimally) maximize the number of color classes covered (exactly one point from each color class) by Q in polynomial time.

2 Covering Color Classes

Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of n color classes, where each color class C_i is a pair of points $\{p_i, \overline{p_i}\}$ on the x-axis. We call an interval on the x-axis that contains at most one point from each color class a conflict free interval (CF-interval). A main goal in this section is to prove that the following problem is NP-hard; additionally, we give a 2-approximation.

Problem 1 Covering color classes with CF-intervals. *Find a minimum-cardinality set $\mathcal{I}$ of* arbitrary *CF-intervals, such that at least one point from each color class is covered by an interval in $\mathcal{I}$.*

Before presenting the proof, we prove that the problem in which one has to pick the covering CF-intervals from a given set of CF-intervals is NP-hard. We then use this result in our proof for Problem 1, together with an auxiliary result stated as Theorem 2 below.

2.1 Covering with a given set of CF-intervals

We prove that the following problem is NP-hard.

Problem 2 Covering color classes with a given set of CF-intervals. *Given a set $\mathcal{I}$ of CF-intervals, find a minimum-cardinality set $\mathcal{I}' \subseteq \mathcal{I}$ (if it exists), such that at least one point from each color class is covered by an interval in $\mathcal{I}'$.*

We describe a reduction from vertex cover. A *vertex cover* of a graph G is a subset of the vertices of G, such that each edge of G is incident to at least one vertex of the subset. Given a positive integer k, determining whether there exists a vertex cover of size k is an NP-complete problem [9]. Let $G = (V, E)$ be a graph, where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. We construct a set P of point color classes and a set $\mathcal{I}$ of CF-intervals, such that G has a vertex cover of size k if and only if there exists a subset $\mathcal{I}' \subseteq \mathcal{I}$ of size k that covers at least one point from each color class.

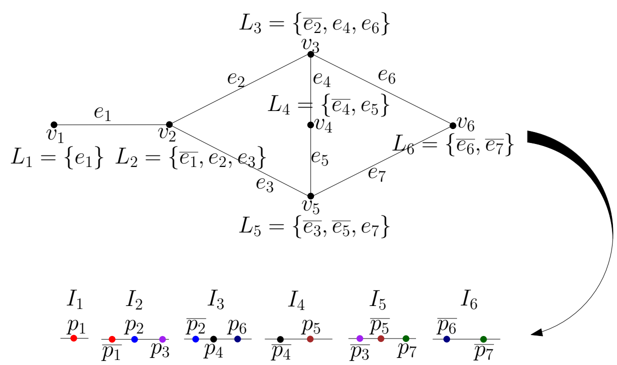

Figure 1: Reduction from vertex cover.

For each vertex v_i create an initially empty set L_i. For each edge $e_k = \{v_i, v_j\}$, where $i < j$, add e_k to L_i and $\overline{e_k}$ to L_j. Now, draw n disjoint intervals on the x-axis, one per set, such that interval I_{i+1} is to the right of interval I_i, $i = 1, \ldots, n-1$. Moreover, for each set L_i, draw $|L_i|$ arbitrary points on the interval I_i as follows. For each element in L_i, if it is of the form e_j, then add the point p_j to I_i, and if it is of the form $\overline{e_j}$, then add the point $\overline{p_j}$ to I_i. Finally, set $P = \{\{p_1, \overline{p_1}\}, \ldots, \{p_m, \overline{p_m}\}\}$ and $\mathcal{I} = \{I_1, \ldots, I_n\}$. See Figure 1 for an illustration.

It is easy to see that G has a vertex cover of size k if and only if there exist k intervals in $\mathcal{I}$ which together cover at least one point from each color class in P. Hence we have the following theorem.

Theorem 1 *Problem 2 is NP-hard.*

2.2 Covering with arbitrary CF-intervals

In order to show that Problem 1 is NP-hard, we first need to prove the following theorem, which says that minimum vertex cover remains NP-hard even when we restrict our attention to highly dense graphs.

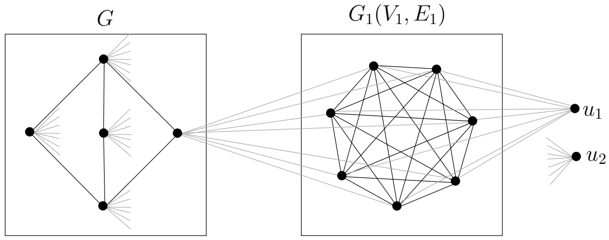

Figure 2: The graph G'.

Theorem 2 (Min vertex cover in dense graphs) *Finding a minimum vertex cover of a graph in which the degree of each vertex is at least $\frac{n}{2}$ is NP-hard, where n is the number of vertices in the graph.*

Proof. Let $G = (V, E)$ be any graph. We construct a new graph $G' = (V', E')$ in which the degree of each vertex is at least $\frac{|V'|}{2}$, and show that one can immediately obtain a minimum vertex cover of G from a minimum vertex cover of G' (and vice versa).

Let $G_1 = (V_1, E_1)$ be the complete graph of $|V| + 2$ vertices. We construct G' as follows. Set $V' = V \cup V_1 \cup \{u_1, u_2\}$, where u_1, u_2 are two new vertices. Set $E' = E \cup E_1 \cup E_2 \cup E_3$, where $E_2 = V \times V_1$ and $E_3 = V_1 \times \{u_1, u_2\}$ (see Figure 2). Notice that G' has the desired property, i.e., for each $v \in V'$, the degree of v (in G') is at least $\frac{|V'|}{2} = \frac{2|V|+4}{2} = |V| + 2$. (If v comes from V, then $deg_{G'}(v) = deg_G(v) + |V| + 2 \geq |V| + 2$, if v comes from V_1, then $deg_{G'}(v) = deg_{G_1}(v) + |V| + 2 \geq |V| + 2$, and if $v \in \{u_1, u_2\}$, then $deg_{G'}(v) = |V_1| = |V| + 2$.)

We now claim that given a minimum vertex cover of G', one can immediately obtain a minimum vertex cover of G, and vice versa. Let V^* be a minimum vertex cover of G'. We first show that $V_1 \subseteq V^*$. Since G' contains the complete graph G_1 of size $|V| + 2$, any minimum vertex cover of G' must include at least $|V| + 1$ vertices of V_1. If one of V_1's vertices, v, is not in V^*, then both u_1 and u_2 are necessarily in V^* (to cover the edges $\{v, u_1\}, \{v, u_2\}$). But, if so, V^* is not a minimum vertex cover, since $V^* \setminus \{u_1, u_2\} \cup \{v\}$ is also a vertex cover of G'. We conclude that $V_1 \subseteq V^*$. Notice that V_1 covers all the edges in E' except for the edges in E. Thus, the rest of the vertices in V^* consist of a minimum vertex cover of G. In other words, $V^* \cap V$ is a minimum vertex cover of G.

On the other hand, let $\tilde{V}$ be a minimum vertex cover of G, then $V_1 \cup \tilde{V}$ is a minimum vertex cover of G'. (Since, as shown above, V_1 is contained in any minimum vertex cover of G', and in order to cover the remaining uncovered edges, we need a minimum vertex cover of G.) □

Corollary 3 *Finding a minimum vertex cover of a graph $G = (V, E)$ in which the degree of each vertex is at least $\varepsilon|V|$, where $0 < \epsilon < 1$, is NP-hard.*

Proof. Similar to the proof of Theorem 2. □

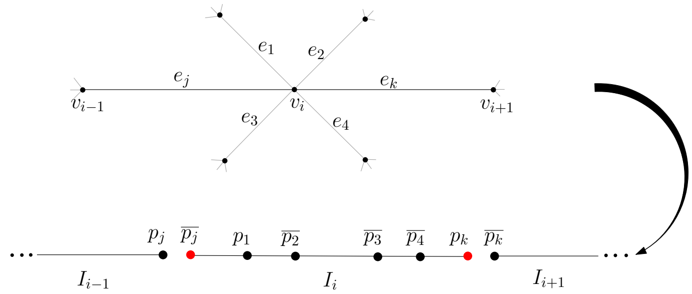

Figure 3: Illustration of Theorem 4.

We are now ready to prove that Problem 1 is NP-hard. We describe a reduction from minimum vertex cover in dense graphs (see Theorem 2 above). Let $G = (V, E)$ be any graph in which the degree of each vertex is at least $\frac{n}{2}$, where $n = |V|$. By Dirac's theorem [5] (or Ore's theorem [11]), G contains a Hamiltonian cycle; moreover, Palmer [12] presented a simple and efficient algorithm for computing such a cycle, under the conditions of Ore's theorem.

Let $v_1, v_2, \ldots, v_n, v_1$ be a Hamiltonian cycle in G. As for Problem 2, we construct a set P of point color classes. For each vertex $v_i \in V$, we construct a set L_i as follows. For each edge $e_k = \{v_i, v_j\}$ adjacent to v_i, we add e_k (resp., $\overline{e_k}$) to L_i, if $i < j$ (resp., $j < i$). We now draw n disjoint intervals on the x-axis, such that interval I_i corresponds to set L_i and precedes interval I_{i+1} (for $i < n$). We draw $|L_i|$ points in I_i as follows. Let $e_j = \{v_{i-1}, v_i\}$ and $e_k = \{v_i, v_{i+1}\}$. Then $\overline{e_j}, e_k \in L_i$. Place a point $\overline{p_j}$ corresponding to $\overline{e_j}$ at the left endpoint of I_i and place a point p_k corresponding to e_k at the right endpoint of I_i. In addition, place a point anywhere in the interior of I_i, for each of the other elements in L_i. For example, in Figure 3 e_j and e_k are the edges connecting v_i to v_{i-1} and to v_{i+1}, respectively, and e_1, e_2, e_3, e_4 are the other edges incident to v_i. The corresponding interval representation is shown in Figure 3.

Now, set $P = \{\{p_1, \overline{p_1}\}, \{p_2, \overline{p_2}\}, \ldots\}$ and $\mathcal{I} = \{I_1, \ldots, I_n\}$. Observe that I_i is conflict free (by construction), for $i = 1, \ldots, n$. Moreover, any other CF-interval is necessarily contained in one of the intervals already in $\mathcal{I}$ (since any interval that covers the right endpoint of I_i and the left endpoint of I_{i+1} is not conflict free). Thus, one might as well pick intervals from $\mathcal{I}$ when covering the color classes of P with a minimum number of arbitrary CF-intervals. But, by Theorem 1 this is NP-hard. Hence we have the following theorem.

Theorem 4 *Problem 1 is NP-hard.*

A 4-approximation algorithm for Problem 2. Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of point color classes (pairs, $C_i = \{p_i, \overline{p_i}\}$) on the set of points $\mathcal{P} = \bigcup_i C_i$. We assume there exists $\mathcal{I}' \subseteq \mathcal{I}$ such that $\mathcal{I}'$ covers at least one point from each color class and we provide a 4-

approximation algorithm for covering P with the fewest number of CF-intervals. For a given $p \in \mathcal{P}$, let $I_p \in \mathcal{I}$ be a CF-interval (if it exists) that covers p and extends farthest to the right among all intervals that cover p. Let $I_p^{(r)} \subseteq I_p$ be the subinterval of I_p that contains p and all points to the right of p.

Input: $P = \{C_1, C_2, \ldots, C_n\}$, a set of point color classes and $\mathcal{I}$, a set of CF-intervals.
Output: A subset $\mathcal{I}' \subseteq \mathcal{I}$ covering at least one point from each color class.
$\mathcal{T} = \emptyset$
while *there exists* $p \in \mathcal{P}$ *such that* p *is uncovered in* $\mathcal{T}$ *and there exists* $I \in \mathcal{I}$ *such that* $p \in I$ **do**
  Let p be the leftmost uncovered point in $\mathcal{P}$ that is contained in some interval in $\mathcal{I}$.
  $\mathcal{T} \leftarrow \mathcal{T} \cup I_p^{(r)}$
end
Compute a subset of intervals $\mathcal{T}$ to cover at least one point of each of the C_i's, using a low-frequency set cover approximation algorithm.
Let $\mathcal{I}'$ be the set of intervals $I_p \in \mathcal{I}$ corresponding to each $I_p^{(r)}$ of $\mathcal{T}$ in the resulting cover.

Algorithm 1: An algorithm for Problem 2.

Lemma 5 $|\mathcal{I}'| \leq 4|OPT|$.

Proof. Consider the set $\mathcal{T}$ of intervals at the end of the while loop. Let $OPT_{\mathcal{T}} \subseteq \mathcal{T}$ be an optimal set cover of the C_i's. First we claim that $|OPT_{\mathcal{T}}| \leq 2|OPT|$. Consider the leftmost point p in an arbitrary interval A of OPT. By the construction of Algorithm 1, we know that there must exist an interval $T \in \mathcal{T}$ that contains p. If there exists a point that is covered by A and not covered by T, then let q be the leftmost such point. We know there exists an interval $I_q^{(r)} \in \mathcal{T}$ that starts at q and extends at least as far to the right as does A. Thus, for any $A \in OPT$, there exist at most two intervals in T, the union of which entirely contains A.

Observe that since each newly added interval to $\mathcal{T}$ cannot contain a previously covered point, then, at the end of the while loop, each $p \in \mathcal{P}$ is contained in at most one interval of $\mathcal{T}$; thus, each pair C_i is covered by at most two intervals of $\mathcal{T}$ (one covering p_i, one covering $\overline{p_i}$). Therefore, we are approximating a low-frequency (at most 2) set cover instance, for which LP relaxation gives a 2-approximation [14] (pp. 119-120). Hence, we have $|\mathcal{I}'| \leq 2|OPT_{\mathcal{T}}| \leq 4|OPT|$. (For color classes of size at most c, we obtain a $2c$-approximation.) □

A 2-approximation algorithm for Problem 1. Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of point color classes (pairs, $C_i = \{p_i, \overline{p_i}\}$) on the set of points $\mathcal{P} = \bigcup_i C_i$. We provide a simple 2-approximation algorithm for covering P with arbitrary CF-intervals. For any point $p \in \mathcal{P}$, denote the maximal CF-interval starting at p and ending at a point of $\mathcal{P}$ to the right of p (or at p) by $I_{\max}(p)$.

Input: $P = \{C_1, C_2, \ldots, C_n\}$, a set of point color classes.
Output: A set $\mathcal{I}$ of CF-intervals.
$\mathcal{I} = \emptyset$
while $\mathcal{P} \neq \emptyset$ **do**
  Let p be the leftmost point in $\mathcal{P}$
  $\mathcal{I} \leftarrow \mathcal{I} \cup I_{\max}(p)$
  For each point of $\mathcal{P}$ that lies in $I_{\max}(p)$, remove it and its twin point from $\mathcal{P}$
end

Algorithm 2: A greedy algorithm for Problem 1.

Consider the set $\mathcal{I}$ computed by Algorithm 2. Clearly, $\mathcal{I}$ is a set of (disjoint) CF-intervals, such that at least one point from each color class is covered by the intervals of $\mathcal{I}$. It remains to prove that $\mathcal{I}$ is a 2-approximation of OPT, where OPT denotes any optimal solution.

Lemma 6 $|\mathcal{I}| \leq 2|OPT|$.

Proof. For any two points x and y, let $[x, y]$ (resp., (x, y)) denote the closed (resp., open) interval with endpoints x and y. Let $[p_a, p_b]$ and $[p_c, p_d]$ be two consecutive intervals in $\mathcal{I}$. Observe that since $[p_a, p_b]$ is a maximal CF-interval, there exists a point p_i (resp., $\overline{p_i}$) in $[p_a, p_b]$, such that $\overline{p_i}$ (resp., p_i) is in (p_b, p_c). Therefore any interval in OPT can intersect at most two intervals in $\mathcal{I}$. Moreover, since OPT must cover the color class $C_i = \{p_i, \overline{p_i}\}$, there exists an interval $I \in OPT$, such that $I \cap \{p_i, \overline{p_i}\} \neq \emptyset$. We thus conclude that $|OPT| \geq |\mathcal{I}|/2$. □

3 Two Dimensions

Let $P = \{C_1, C_2, \ldots, C_n\}$ be a set of n color classes in the Euclidean plane. We explore covering problems where exactly one point from each color class must be covered.

3.1 Unit Squares

Problem 3 Covering color classes with arbitrary unit squares. *Let $P = \{C_1, C_2, \ldots, C_n\}$ be in the Euclidean plane and let each color class C_i consist of a vertically or horizontally unit separated pair of points. Find a minimum-cardinality set $\mathcal{S}$ of axis-aligned unit squares (assuming a feasible solution exists), such that exactly one point from each color class is covered by a square in $\mathcal{S}$.*

Theorem 7 *Problem 3 is NP-hard.*

Proof. The reduction is from PLANAR 3-SAT [10], where one is given a formula in conjunctive normal form with at most three literals per clause, with the objective of deciding whether or not the formula is satisfiable. Given variables $\{x_1, x_2, \ldots, x_n\}$ and clauses $\{c_1, c_2, \ldots, c_m\}$, we consider the graph whose nodes are the clauses and variables and whose edges join variable x_i with clause c_j if and only if $x_i \in c_j$ or $\neg x_i \in c_j$. The resulting bipartite graph, G, is planar.

In a manner similar to Fowler et al. [6], in a planar embedding of G we replace all of the edges incident to a variable node with a variable chain that visits the corresponding clauses and returns to the variable node to form a loop. The variable chains consist of a sequence of unit separated pairs (see Figure 4) and are designed in such a way that any minimum cardinality solution will either cover $\{\overline{a_{i+k}}, a_{i+k+1}$: k is even$\}$ or $\{\overline{a_{i+k}}, a_{i+k+1}$: k is odd$\}$. That is, for a given variable chain, either all blue unit squares or all red unit squares will be used. Using red (resp. blue) squares for variable x_i is equivalent to setting this variable to TRUE (resp. FALSE). Using planarity of the graph embedding, no two variable chains intersect, and any two points from different chains are spaced at least unit distance apart.

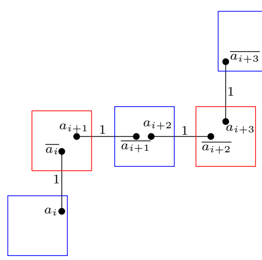

Figure 4: Variable chain.

Clause c_i consists of a single (green) pair (see Figure 5). If c_i evaluates to FALSE, then a square that is not associated with any variable loop will be needed to cover c_i. If c_i evaluates to TRUE, then a point from c_i can be covered by a square from an incoming loop whose literal evaluates to TRUE.

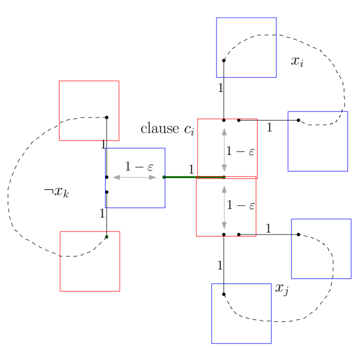

Figure 5: Clause gadget.

Let r_i be the number of pairs used in variable chain $i, 1 \leq i \leq 3m$. We design the variable chains so that r_i is even for all i. Let $r = \sum_i r_i$. It is now apparent that there exists a satisfying truth assignment in PLANAR 3-SAT if and only if a minimum cardinality covering with unit squares uses $\frac{r}{2}$ squares. □

Remark: If P is on a line and pairs are unit separated, we can minimize the number of unit intervals used in a complete cover (assuming a solution exists) in polynomial time using dynamic programming.

A 6-approximation algorithm. We lay out a grid with unit dimensions on top of our point set P and two-color the cells of the grid red and black in the style of a checkerboard. We say that a cell is occupied if it contains a point in P. Let R be the set of occupied red cells and B the set of occupied black cells. As a solution, we use the set of smaller cardinality.

Lemma 8 $\min\{|R|, |B|\} \leq 6|OPT|$.

Proof. Suppose w.l.o.g that $\min\{|R|, |B|\} = |R|$. Note that R is a feasible solution because any two points of a color class are unit separated either vertically or horizontally, thus one of the two points must occupy a red cell and the other must occupy a black cell. Therefore, R covers all color classes of points and no two points from the same color class are covered by R.

Now we claim that in the optimal solution, OPT, at least $\frac{1}{12}(|R| + |B|)$ unit squares are used. An arbitrary unit square, s, used in OPT stabs at most four cells of the checkerboard. These four cells are adjacent to at most eight other cells in total, each of which can be occupied by the pair of one of the points covered by s. Thus, at most 12 occupied cells of the checkerboard can be accounted for by any unit square used in OPT.

Combining the fact that $\min\{|R|, |B|\} \leq \frac{1}{2}(|R|+|B|)$ and $OPT \geq \frac{1}{12}(|R|+|B|)$, we have that $\min\{|R|, |B|\} \leq 6|OPT|$. □

3.2 Covering with a Convex Polygon

Problem 4 *Let $P = \{C_1, C_2, \ldots, C_n\}$ be in the Euclidean plane and let each color class C_i consist of either a pair or a triple of points. Decide whether or not there exists a convex polygon Q such that Q contains exactly one point from each color class.*

Problem 5 *Let $P = \{C_1, C_2, \ldots, C_n\}$ be in the Euclidean plane and let each color class C_i consist of a pair of points. Maximize the number of color classes covered by a convex polygon Q such that Q contains exactly one point from each covered color class.*

Theorem 9 *Problem 4 is NP-complete.*

Proof. Problem 4 is clearly in NP because we can check whether or not polygon Q is convex and whether or not Q contains exactly one point from each color class in polynomial time. We present a reduction from EXACTLY 1-IN-3-SAT, where one is given a formula in conjunctive normal form with at most three literals per clause, with the objective of deciding whether or not the formula is satisfiable. In a satisfying assignment, every clause must contain exactly one TRUE literal.

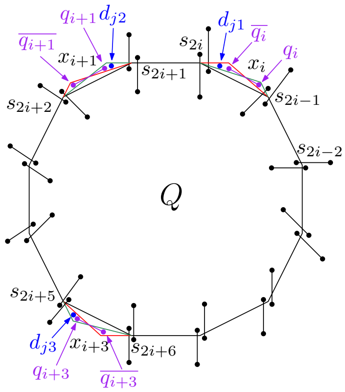

Figure 6: Construction of hardness for Problem 4.

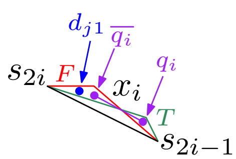

Figure 7: Close-up of variable gadget for Problem 4.

Given variables $\{x_1, x_2, \ldots, x_n\}$ and clauses $\{c_1, c_2, \ldots, c_m\}$, we start by considering $2n$ points, $S = \{s_1, s_2, \ldots, s_{2n}\}$, in the position of a regular $2n$-gon. These $2n$ points are not part of any color class; we use them to help explain the construction. We place two pairs of points around each point of S in such a way that convex polygon Q must have vertices at each point of S (see Figure 6). We create a variable gadget x_i in between points s_{2i-1} and s_{2i} for $1 \leq i \leq n$. Each variable gadget consists of color class that is a pair of points $\{q_i, \overline{q_i}\}$, $1 \leq i \leq n$ (see Figure 7). We place $\{q_i, \overline{q_i}\}$ so that Q can be expanded to cover either q_i (green lines in Figure 7) or $\overline{q_i}$ (red lines in Figure 7), while remaining convex. Setting x_i to TRUE (resp. FALSE) corresponds to expanding Q to cover q_i (resp. $\overline{q_i}$). If x_i (resp. $\neg x_i$) appears in clause c_j, a point from a color class that contains triple $\{d_{j1}, d_{j2}, d_{j3}\}$ will appear in the expansion of Q that covers q_i (resp. $\overline{q_i}$), and not in the expansion of Q that covers $\overline{q_i}$ (resp. q_i). It is now apparent that there exists a satisfying truth assignment in EXACTLY 1-IN-3-SAT if and only if convex polygon Q covers exactly one point from each color class. $\square$

Theorem 10 *Problem 5 is NP-hard.*

Proof. The reduction is from MAX EXACTLY 1-IN-2-SAT where each clause has at most two literals and the objective is to maximize the number of clauses that evaluate to TRUE. A clause evaluates to TRUE if and only if it contains exactly one TRUE literal. Using the same construction as in Problem 4, it is easy to see that maximizing the number of TRUE clauses is equivalent to maximizing the number of color classes covered. $\square$

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