

# Folding Polyominoes into (Poly)Cubes\*

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## Abstract

We study the problem of folding a given polyomino  $S$  into a polycube  $C$  under different folding models, allowing faces of  $C$  to be covered multiple times.

## 1 Introduction

When can a polyomino  $S$  be folded into a polycube  $C$ ? This problem has been considered by Abel et al. [1] and Aloupis et al. [2], but with the restriction that there must be a one-to-one mapping between the unit squares of  $S$  and the faces of  $C$ . We allow polycube faces to be covered multiple times, only requiring  $C$  to be covered by  $S$ . We show that different sets of allowed folding angles give distinct variations from each other. We characterize polyominoes that can fold to a single cube, count foldings of polyominoes of different orders into cubes, and investigate the complexity of finding foldings into higher-order polycubes.

## 2 Notation

A *polyomino*  $S$  is a 2D polygon formed by a union of  $|S| = n$  *unit squares* on the square lattice connected edge-to-edge. Not all edge-to-edge connections of the  $n$  unit squares must be used for the polyomino, that is we allow “cuts” on the lattice. A polyomino is a *tree shape* if the dual graph of its unit squares is a tree. A *polycube*  $C$  is a connected 3D polyhedron formed by a union of

*unit cubes* on the cubic lattice connected face-to-face. If  $C$  is a rectangular parallelepiped, we refer to its size by its exterior dimensions, e.g., a  $2 \times 2 \times 1$ -polycube.

We study the problem of folding a given  $S$  into a given  $C$ , allowing *axis-aligned*  $+90^\circ$  and  $+180^\circ$  mountain folds,  $-90^\circ$  and  $-180^\circ$  valley folds, folds of *any* degree, *diagonal folds* through opposite corners of a square, and *half-grid folds* that bisect a unit square in an axis-parallel fashion.

A face of  $S$  is an *interior face* of  $C$  if it is not flat folded on any of the outer faces of  $C$ ; see Fig. 3(a) and (b) for examples. A folding model  $\mathcal{F}$  specifies a subset of  $\mathcal{F} = \{\text{grid: } +90^\circ, -90^\circ, +180^\circ, -180^\circ, \text{any}^\circ; \text{interior faces; diagonal; half-grid}\}$  as allowable folds.

## 3 Folding hierarchy

We say that model  $\mathcal{F}_x$  is stronger than  $\mathcal{F}_y$  ( $\mathcal{F}_x \geq \mathcal{F}_y$ ) if for all polyomino-polycube pairs  $\{S, C\}$  such that  $S$  folds into  $C$  in  $\mathcal{F}_y$ ,  $S$  also folds into  $C$  in  $\mathcal{F}_x$ . If there also exists a pair  $\{S', C'\}$  such that  $S'$  folds into  $C'$  in  $\mathcal{F}_x$ , but not in  $\mathcal{F}_y$ , then  $\mathcal{F}_x$  is strictly stronger than  $\mathcal{F}_y$  ( $\mathcal{F}_x > \mathcal{F}_y$ ). The relation ‘ $\geq$ ’ satisfies the properties of reflexivity, transitivity and antisymmetry, therefore it defines a partial order on the set of folding models. Fig. 1 shows the resulting hierarchy of the folding models that consist of combinations of the following folds:  $\{\text{grid: } +90^\circ, -90^\circ, +180^\circ, -180^\circ, \text{any}^\circ; \text{interior faces}\}$ .

Integrating diagonal and half-grid folds (which are omitted from this section) can result in stronger models: a  $1 \times 7$  polyomino can be folded into a unit cube  $C$  in model  $\{\text{grid: } \pm 90^\circ/180^\circ; \text{diagonal}\}$ , but not in  $\{\text{grid: } \text{any}^\circ; \text{interior}\}$  (the strongest model from Fig. 1); the example from Fig. 4(b)–(c) shows that  $\mathcal{F}_{all} = \{\text{grid: } \text{any}^\circ; \text{interior faces; diagonal; half-grid}\}$  is strictly stronger than  $\mathcal{F} = \{\text{grid: } \text{any}^\circ; \text{interior faces; diagonal}\}$ . In addition, Lemma 3 still holds for  $\mathcal{F} = \{\text{grid: } \pm 90^\circ/180^\circ; \text{interior faces; diagonal; half-grid}\}$ .

The following establishes the relationships between models presented in Fig. 1.

**Theorem 1** *The folding models consisting of combinations of the following folds  $\{+90^\circ, -90^\circ, +180^\circ, -180^\circ, \text{arbitrary degree folds, interior folds}\}$  have the mutual relations presented in Fig. 1. In particular, these mutual relations hold for polyominoes without holes.*

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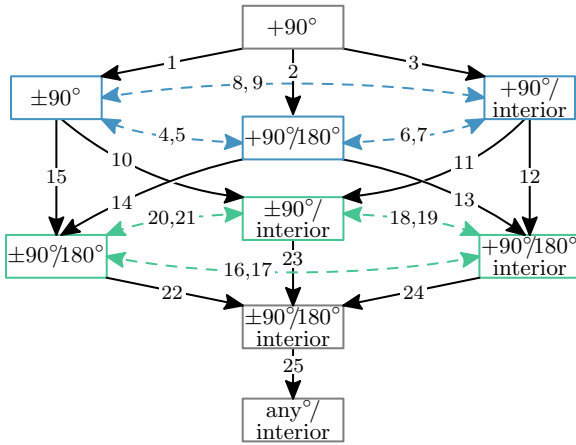


Figure 1: Hierarchy of fold operations. A black arrow from  $\mathcal{F}_x$  to  $\mathcal{F}_y$  indicates that  $\mathcal{F}_y > \mathcal{F}_x$ . Blue and green arrows indicate incomparable models.

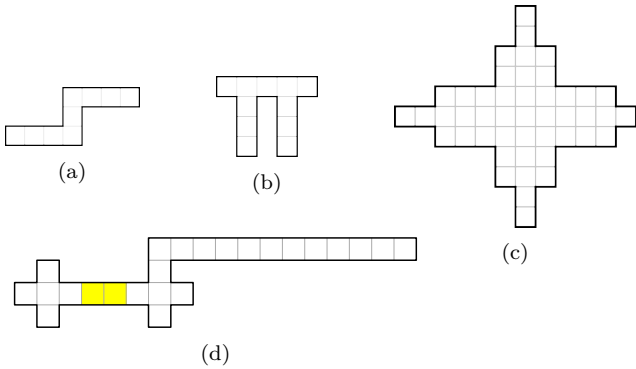


Figure 2: Examples for Thm. 1, where  $C$  is (a) a unit cube, (b) a  $2 \times 1 \times 1$  polycube, (c) a  $3 \times 3 \times 2$  polycube with a 1-cube hole centered in a  $3 \times 3$  face, and (d) a 5-cube cross.

**Proof.** All cases can be shown using the polyomino-polycube pairs from Fig. 2. They fold with: (a)  $+90^\circ/180^\circ$  folds, (b)  $+90^\circ$  and interior faces (see Lemma 3), (c)  $\pm 90^\circ$  folds, (d)  $+90^\circ$ ,  $180^\circ$  and interior faces.

- Ex. (a) shows relations 2, 4, 7, 12, 15, 19, 20, 23.
- Ex. (b) shows relations 3, 6, 9, 10, 13, 16, 21, 22.
- Ex. (c) shows relations 1, 5, 8, 11, 14, 17, 18, 24.
- Ex. (d) shows relation 25.

We will detail as an example the proof for relation 25; the others are left to the reader. The claim is that  $\mathcal{F}_9 = \{\text{grid: any}^\circ; \text{interior faces}\}$  is strictly stronger than  $\mathcal{F}_8 = \{\text{grid: } \pm 90^\circ/180^\circ; \text{interior faces}\}$ . Any polyomino  $S$  that folds into polycube  $C$  in  $\mathcal{F}_8$  also folds into  $C$  in  $\mathcal{F}_9$ . To prove a strict relation, we are left to show that there exists some polyomino  $S'$  that folds into some polycube  $C'$  in  $\mathcal{F}_9$ , but that does not fold into  $C'$  in  $\mathcal{F}_8$ . Let  $C'$  consist of five cubes forming a cross, and let  $S'$  be as in Fig. 2(d). Assume that  $S'$  can be folded into  $C'$  in folding model  $\mathcal{F}_8$ .  $|S'| = 24$ , while  $C'$  has 22 square faces on its surface. Therefore, 22 out of the 24 squares

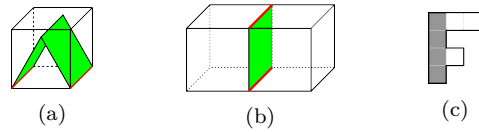


Figure 3: (a)/(b) Examples for interior faces shown in green. They connect to the red cube edges. (c) Shape  $S$  from the proof of Lemma 2.

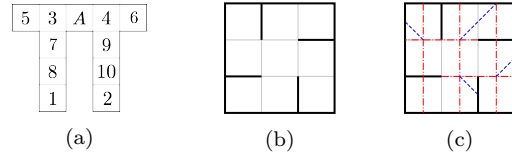


Figure 4: (a) Shape  $S$  needs interior faces to be folded into a  $2 \times 1 \times 1$ -cube. (b) Shape  $S$  does not fold to a unit cube  $C$ , if we allow  $\pm 90^\circ/180^\circ$  and diagonal folds. However, if we allow half-grid folds,  $S$  does fold into  $C$ , the mountain (red) and valley folds (blue) are shown in (c).

of  $S'$  will be the faces of  $C'$  when folded. Consider the  $12 \times 1$  sub-polyomino of  $S'$ . When folded, it has to form the “walls” of the cross  $C'$ , otherwise this strip can form not more than eight faces of the cross (looping around one of the  $3 \times 1 \times 1$  sub-polycubes). It is straightforward to see now, that in  $\mathcal{F}_8$  the two yellow squares prevent  $S'$  from folding into  $C'$ . While in  $\mathcal{F}_9$  the two yellow squares can form interior faces with a  $60^\circ$  interior fold. Therefore,  $\mathcal{F}_8 < \mathcal{F}_9$ .  $\square$

**Conjecture 1** *Theorem 1 holds for tree shapes.*

**Lemma 2** *There exist tree shapes  $S$  that need both mountain and valley folds to cover a unit cube.*

**Proof.** The shape  $S$  from Fig. 3(c) does not fold into a unit cube  $C$  with only valley folds: the four unit squares in the left column (gray) fold to a ring of size four, and the two flaps can only cover one of the two remaining cube faces. But with both mountain and valley folds  $S$  can be folded into  $C$ , by using a  $180^\circ$  fold between the first column and the longer flap.  $\square$

**Lemma 3** *Folding the tree shape  $S$  from Fig. 2(b) into a  $2 \times 1 \times 1$ -cube  $C$  with  $\mathcal{F} = \{\text{grid: } \pm 90^\circ/180^\circ; \text{interior faces}\}$  requires interior faces. (Any of the faces  $A$ , 8 or 10 from Fig. 4(a) can be the interior face.)*

**Proof.** We label the faces of  $S$  as shown in Fig. 4(a). The case analysis below shows that a folding without interior faces does not exist:

If  $A$  covers one of the  $1 \times 1$  faces of  $C$ , face 1 or 2 is needed for the opposite  $1 \times 1$  face. By symmetry suppose face 1 covers that side. Thus, 3 needs to be folded on top of  $A$ . Face 4 folds either on 5 or  $A$ . So, we doubled two

faces. But as  $|S| = 11$  and  $C$  has 10 faces, the remaining faces are not enough.

If 3,  $A$ , and 4 all cover parts of  $1 \times 2$  faces of  $C$ , then the row of 5, 3,  $A$ , 4, 6 maps to only four  $1 \times 2$  faces with at least one overlap. But then the  $1 \times 1$  face adjacent to  $A$  cannot be covered by 7, 8, 9, 10, 1 or 2 without doubling on 3 or 4—again more than one overlap.

If  $A$  covers part of a  $1 \times 2$  face of  $C$  and one of 3 or 4 (without loss of generality 3) covers the adjacent  $1 \times 1$  face, then the only way to cover the two squares that are adjacent to  $A$  in the polycube but not in the polyomino is to double 4 with  $A$  and wrap column 4–9–10–2 around the polycube; but then 6 must also be doubled, again leading to more than one overlapping pair.  $\square$

#### 4 Polyominoes that fold into a cube

In this section we characterize all polyominoes that can be folded into a unit cube and all tree shaped polyominoes that are a subset of a  $2 \times n$  or  $3 \times n$  strip that can be folded into a unit cube using arbitrary grid folds.

**Theorem 4** Consider a polyomino  $S$  of size  $|S| = n$  and a unit cube  $C$  under a folding model  $\mathcal{F} = \{\text{grid}; \text{any}^\circ; \text{diagonal}; \text{half-grid}\}$ , such that each face of  $C$  has to be covered by a full unit square of  $S$ . Then  $n \geq 10$  is the best possible universal bound, i.e., there is a polyomino of size  $n = 9$  that cannot be folded into  $C$ , while all polyominoes with  $n \geq 10$  can be folded.

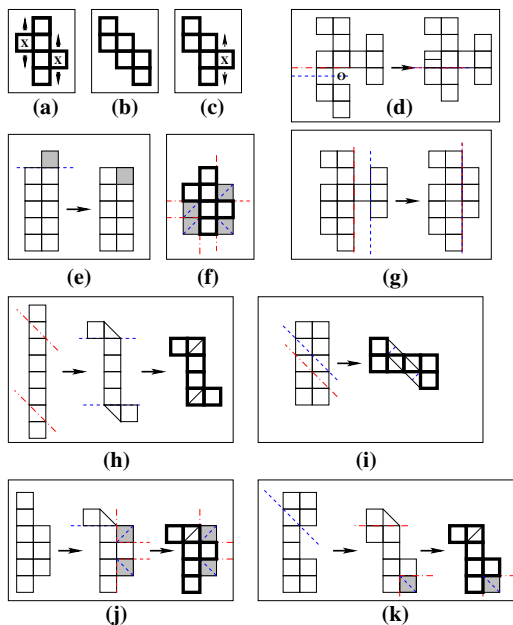


Figure 5: Proof details for Theorem 4. Unit squares marked with "X" can be located at an arbitrary height. For the two adjacent corner reductions in the left of (f), fold away the upper shaded corner before folding away the lower one.

**Proof.** We consider a bounding box of  $S$  of size  $X \times Y$ , with  $X \leq Y$ . The unit squares are arranged in *columns* and *rows*, indexed  $1, \dots, X$  and  $1, \dots, Y$ , with  $n_i$  unit squares  $S$  in column  $i$ , and  $m_j$  unit squares in row  $j$ .

For the lower bound see Fig. 4(b). Note that if we allow half-grid folds without requiring faces of  $C$  to be covered by full unit squares of  $S$ , we can turn this shape  $S$  into a unit cube; see Fig. 4(c).

For the upper bound  $n \geq 10$ , we start by identifying several *target polyominoes* shown in Fig. 5(a)–(c). Each can be folded into a cube using only grid folds:

- A  $1+4+1$  polyomino, composed of one contiguous column of four unit square, with one more unit square on either side, at an arbitrary height.
- A  $2-2-2$  polyomino, composed of three (vertical) pairs attached in the specific manner shown.
- A  $2-3+1$  polyomino, composed of a (vertical) pair and triple attached in the specific manner shown, with one more unit square at an arbitrary height.

See Fig. 5(d)–(g) for the following. If  $n_i$  is the maximum number of unit squares in any column (say, in  $i$ ), we can apply a *connectivity reduction* (d) by using (horizontal) half-grid folds to convert  $S$  into a polyomino  $S'$  in which these  $n_i$  unit squares form a contiguous set, while leaving at least one unit square in each previously occupied column. A *number reduction* (e) lets us fold away extra unit squares for  $n > 10$ . *Corner reductions* (f) fold away unnecessary unit squares (or half-squares) in target shapes by using diagonal folds when turning them into a unit cube. Finally, *width* and *height reductions* (g) fold over whole columns or rows of  $S$  onto each other, producing a connected polyomino with a smaller total number of columns.

Now consider a case distinction over  $X$ ; see Fig. 5(h)–Fig. 6(t). For  $X = 1$ , the claim is obvious, as we can reduce  $S$  to a  $1+4+1$  target; see Fig. 5(h). For  $X = 2$ , note that  $Y \geq 5$  and assume that  $n_1 \geq n_2$ . If  $n_1 \geq 8$ , a width reduction yields the case  $X = 1$ , so assume  $n_1 \leq 7$ , and therefore  $n_2 \geq 3$ . By a number reduction, we can assume  $n_2 \leq 5$ . If  $S$  is a  $2 \times 5$  polyomino, we can make use of a  $1+4+1$  polyomino with corner reductions; see Fig. 5(i). If  $n_1 > n_2$ , we have  $n_1 \geq 6$ . Because  $S$  is connected, any two units squares in column 1 must be connected via column 2, requiring at least three unit squares; because of  $n_2 \leq 5$ , we conclude that column 1 contains at most two connected components of unit squares. Thus, at most one connectivity reduction makes column 1 connected, with  $n'_2 \in \{n_2 - 1, n_2\}$  unit squares in column 2. Possibly using height reduction, we get a connected polyomino  $S''$  with vertical size six, six unit squares in column 1, and  $n''_2 \in \{1, \dots, 4\}$  unit squares in column 2. For  $n''_2 \in \{1, \dots, 3\}$ , there is a reduction to target shape  $1+4+1$ ; see Fig. 5(j). For  $n''_2 = 4$ , a similar reduction exists; see Fig. 5(k). This leaves  $n_1 = 5$ ,

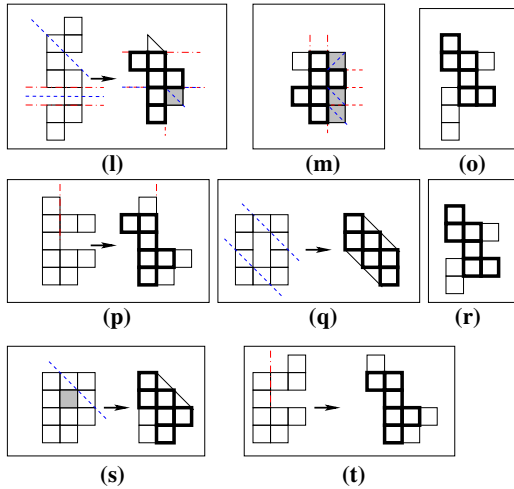


Figure 6: Proof details for Theorem 4.

and thus  $n_2 = 5$ , without  $S$  being a  $2 \times 5$  polyomino. As shown in Fig. 6(l), this maps to a 1+4+1 target shape, by folding a unit square from column 2 that extends beyond the vertical range of the unit squares in column 1 over to column 0.

For  $X = 3$  and  $n_2 \geq 4$ , we can use connectivity, height and width reductions to obtain a new polyomino  $S'$  that has height four, a connected set of  $n'_2 = 4$  unit squares in column 2 and  $1 \leq n'_1 \leq 4$ , as well as  $1 \leq n'_3 \leq 4$  unit squares in columns 1 and 3. This easily converts to a 1+4+1 shape, possibly with corner reductions; see Fig. 6(m). Therefore, assume  $n_2 \leq 3$  and (w.l.o.g.)  $n_1 \geq n_3$ , implying  $n_1 \geq 4$ . If  $n_1 \geq 5$  and column 2 is connected, then  $S$  contains a 2-3+1 target shape, see Fig. 6(o); if column 2 is disconnected, we can use a vertical fold to flip one unit square from column 3 to column 0, obtaining a 1+4+1 target shape, see Fig. 6(p). As a consequence, we are left with  $n_1 = 4$ ,  $2 \leq n_2 \leq 3$ ,  $3 \leq n_3 \leq 4$ ,  $n_2 + n_3 = 6$ , possibly after folding away an extra unit square in column 3 in case of  $n_2 = 3, n_3 = 4$ . If  $n_2 = 2$ , the unit squares in columns 1 and 3 must be connected. For this it is straightforward to check that we can convert  $S$  into a 2-2-2 target polyomino; see Fig. 6(q). Therefore, consider  $n_2 = 3, n_3 = 3$ . This implies that column 1 contains at most two connected sets of unit squares. If there are two, then the unit squares in column 2 must be connected, implying that we can convert  $S$  into a 2-3+1 target shape; see Fig. 6(r). Thus, the four unit squares in column 1 must be connected. If the three unit squares in column 1 are connected, we get a 2-3+1 target shape; see Fig. 6(r). If the unit squares in column 2 are disconnected, but connected by the three unit squares in column 3, we convert this to a 2-2-2 target shape; see Fig. 6(s). This leaves the scenario in which there is a single unit square in column 1 whose removal disconnects the shape; for this we can flip one unit square from column 3 to column 0 in order



Figure 7: A vertical edge in a subset of a  $2 \times n$  strip with possible adjacent vertices for subtrees.

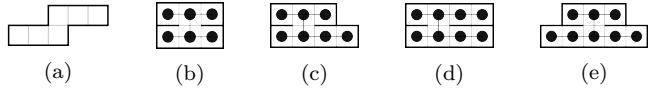


Figure 8: (a) A shape  $S$  that folds into a unit cube. (b)–(e) Shapes  $S$ , only (d) folds into a unit cube.

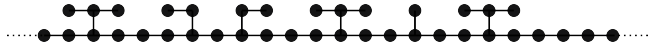


Figure 9: Infinite families that cannot fold into a unit cube.

to create a 1+4+1 target shape; see Fig. 6(t).

For  $X \geq 4$ , we proceed along similar lines. If there is a row or column that contains four unit squares, we can create a 1+4+1; otherwise, a row or column with three unit squares allows generating a 2-3+1. If there is no such row or column, we immediately get a 2-2-2.  $\square$

**Theorem 5** Given a tree shape  $S$ , a unit cube  $C$  and  $\mathcal{F} = \{\text{grid: any}^\circ\}$ .

- (a) If  $S$  is a subset of a  $2 \times n$  strip, then only the infinite families defined by Fig. 9 cannot fold into  $C$ .
- (b) If  $S$  is a subset of a  $3 \times n$  strip, then only the infinite families defined by Fig. 10 cannot fold into  $C$ .

**Proof.** For the subset of a  $2 \times n$  strip consider one vertical edge, as shown in Fig. 7, and the possible subtrees attached at  $A, B, C$  and  $D$ . One such vertical edge has to exist, otherwise the strip is a  $1 \times n$  strip and never folds to a cube. We consider the length of subtrees attached at  $A, B, C$  and  $D$  when folded to the same row as this “docking” unit square to the vertical edge. With slight abuse of notation we refer to these lengths by  $A, B, C$  and  $D$  again.

The first observation is: If  $(A \geq 2 \text{ AND } B \geq 2) \text{ OR } (C \geq 2 \text{ AND } D \geq 2)$ ,  $S$  folds to a cube; see Fig. 8(a). Not included in this categorization are the 4 shapes shown in Fig. 8(b)–(e). Of those only (d) folds into a cube.

Thus, more precisely, we obtain a cube for:

$$\begin{aligned} & \{(A \geq 2 \text{ OR } D \geq 3) \text{ AND } (B \geq 2 \text{ OR } C \geq 3)\} \text{ OR} \\ & \{(A \geq 3 \text{ OR } D \geq 2) \text{ AND } (B \geq 3 \text{ OR } C \geq 2)\} \text{ OR} \\ & \{(A \geq 2 \text{ AND } C \geq 3) \text{ OR } (A \geq 3 \text{ AND } C \geq 2)\} \text{ OR} \\ & \{(B \geq 2 \text{ AND } D \geq 3) \text{ OR } (B \geq 3 \text{ AND } D \geq 2)\} \text{ OR} \\ & \{(A \geq 1 \text{ AND } C \geq 1 \text{ AND } B \geq 2 \text{ AND } D \geq 2)\} \text{ OR} \\ & \{(A \geq 2 \text{ AND } C \geq 2 \text{ AND } B \geq 1 \text{ AND } D \geq 1)\}. \end{aligned}$$

For the subset of a  $3 \times n$  strip: If there are vertical edges adjacent to a  $1 \times n$  strip to two different sides

$n$	free polyominoes	dual trees	foldable with $\pm 90^\circ$	and $\pm 180^\circ$	and diagonal	total
2	1	1				
3	2	2				
4	5	5				
5	12	15				
6	35	54	11	0	0	43
7	108	212	90	24	39	59
8	369	908	571	175	126	36
9	1285	4011	3071	697	233	10
10	4655	18260	15645	2230	385	0
11	17073	84320	77029	6673	618	0
12	63600	394462	374066	19337	1059	0
13	238591	1860872	1803568	55477	1827	0
14	901971	8843896	8682390	158208	3298	0

Table 1: Different ways of folding small polyominoes into a cube.

(above and below), this folds to a cube. Consequently, if we have height three, a long  $1 \times n$  strip cannot be located in the center row. W.l.o.g. let the  $1 \times n$  strip be located in the lowest row. As we have height three, there is at least a height two part which is a subset of the red part in Fig. 10. If there exists another vertical edge of length at least one that is only adjacent to the  $1 \times n$  strip (but not directly to the height two part) we can fold over the height two part and obtain a case from above which can easily be folded to a unit cube. Consequently, only a single vertical subset of length two can be attached, as shown in red in Fig. 10.  $\square$

### 4.1 Enumeration of cube-foldable polyominoes

In this section we present results on folding polyominoes of constant size—consisting of up to 14 unit squares—into a cube. These results have been obtained by exhaustive computer search. For polyominoes whose dual graph contains cycles, we considered all possible dual trees. Thus, we first generated all such dual trees for polyominoes of size up to 14. The third column of Table 1 shows their number, compared to the number of different free polyominoes, given in the second column. In both cases, elements which can be transformed into each other by translation, rotation and/or reflection are counted only once. While the number of different free polyominoes is currently known for shapes of size up to 28 (<http://oeis.org/A000105>), the number of different dual trees was known only for up to 10 elements, see <http://oeis.org/A056841>.

Based on the generated dual trees we checked each of



Figure 10: Infinite families that cannot fold into a unit cube, with at least a height 2 red subset (others are optional).

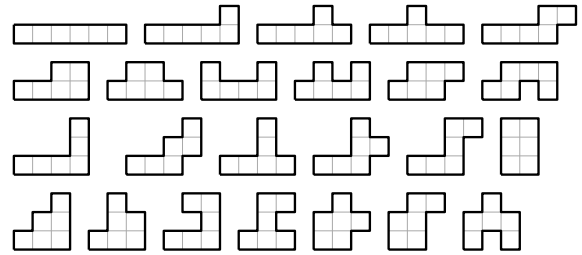


Figure 11: Polyominoes of size 6 that cannot be folded to a cube (for any dual tree).

them whether it can be folded into a unit cube. We did this in three different steps. First, only  $90^\circ$  folds have been allowed. Column 4 of Table 1 shows how many (dual trees of) polyominoes can be folded to a cube this way. It is interesting to observe that while for  $n = 6$  only 11 polyominoes can be folded to a cube, for  $n = 14$  it already works for over 98% of all shapes.

In the second step we tried  $\pm 90^\circ$  and  $\pm 180^\circ$  folds for the remaining dual trees. Table 1 gives in column 5 how many additional cube foldings can be obtained this way. It is interesting to note, that never more than two  $\pm 180^\circ$  folds were needed, if the shape was foldable this way at all. For  $n = 11$  and  $n = 12$  there are each only one example which needs two  $\pm 180^\circ$  folds, and no such examples for  $n \geq 13$  exist, i.e., in that case all foldable examples can be folded with just one  $\pm 180^\circ$  fold.

In the last step we allowed for the remaining dual trees also diagonal folds. Column 6 of Table 1 shows how many additional dual trees can be folded in this case, and the last column gives the number of remaining (non-foldable) dual trees. The most interesting result here is that for  $n \geq 10$  all polyominoes, regardless which dual tree we select for them, can be folded this way. This partially affirms Theorem 4: here we do not allow half-grid folds, but covering a cube face with triangles from diagonal folds. Moreover, all such foldings need at most one such diagonal fold, with the exception of the  $7 \times 1$  strip, which is the only example that needs two diagonal folds.

Figs. 11 and 12 show all polyominoes of size  $n \geq 6$  for which dual trees exist, such that they cannot be

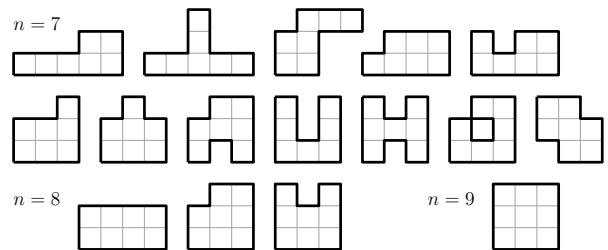


Figure 12: Polyominoes of size 7 to 9 that can be cut into a tree shape that cannot be folded to a cube.

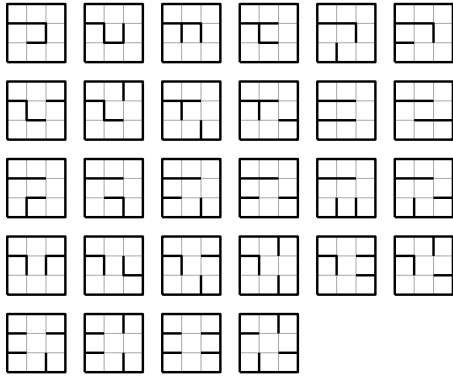


Figure 13: Puzzle: which cuttings of a  $3 \times 3$  square fold to a cube? (For solution see the full version of the paper.)

folded to the unit cube using  $90^\circ$ ,  $180^\circ$ , and diagonal folds. There are 24 such polyominoes with a total of 43 different dual trees for  $n = 6$ , 12 polyominoes with 59 dual trees for  $n = 7$ , 3 polyominoes with 36 dual trees for  $n = 8$ , and one polyomino with 10 dual trees for  $n = 9$ . Note, however, that for many of them there are cuts (i.e., dual trees) such that they can be folded to the cube. For example, the  $3 \times 3$  square has 18 dual trees that can fold to a cube (Fig. 13).

## 5 Dynamic program for trees

**Theorem 6** *Let  $S$  be a tree shape,  $C$  be a polycube with  $O(1)$  cubes, no four squares meeting at an edge, and  $\mathcal{F} = \{\text{grid}: \pm 90^\circ\}$ . Then it is possible in linear time to determine whether  $S$  can fold to  $C$  in folding model  $\mathcal{F}$ .*

**Proof.** (sketch) We choose an arbitrary root of  $S$ ; for each square  $s$  of  $S$  define the subtree of  $s$  to be the tree shape consisting of all squares whose shortest path in  $S$  to the root passes through  $s$ . For a square  $s$  of  $S$ , define a *placement* of  $s$  to be an identification of  $s$  with a surface square of  $C$  together with the subset of the squares of  $C$  covered by squares in the subtree of  $s$ . We use a dynamic program that computes, for each square  $s$  of  $S$ , and each placement of  $s$ , whether there is a folding of the subtree of  $s$  that places  $s$  in the correct position and correctly covers the specified subset. Each square has  $O(1)$  placements, and we can test whether a placement has a valid folding in constant time given the same information for the children of  $s$ . Therefore, the algorithm takes linear time.  $\square$

We have been unable to extend this result to folding models that allow  $180^\circ$  folds, nor to folds with interior faces, nor to polycubes for which four or more squares meet at an edge. The difficulty is that the dynamic program constructs a mapping from the polyomino to the polycube surface (topologically, an *immersion*) but what we actually want to construct is a

three-dimensional embedding of the polyomino without self-intersections, and in general testing whether an immersion can be lifted to a three-dimensional embedding is NP-complete [3]. For  $90^\circ$  folds, a three-dimensional lifting always exists, as can be seen by induction on the number of squares in the tree shape: given a folding of all but one square of the tree shape, there can be nothing blocking the addition of the one remaining square to its neighbor in the tree shape. However, if a pentomino formed by a single row of five squares is given  $+180^\circ$  folds at the two edges incident to its central square, the result cannot be embedded into three dimensional space: one of the two-square flaps will be blocked by the fold from the other flap.

It is tempting to attempt to extend our dynamic program to a fixed-parameter algorithm for non-trees (parameterized by feedback vertex number in the dual graph of the polyomino), by finding an approximate minimum feedback vertex set, trying all placements of the squares in this set, and using dynamic programming on the remaining tree components of the graph. However, the problem of parts of the fold blocking other parts of the fold becomes even more severe in this case, even for  $90^\circ$  folds. Additionally, we must avoid knots and twists in the three-dimensional embedding. These issues make it difficult to extend the dynamic program to the non-tree case.

## 6 Conclusion

Various open problems remain. We gave an example of a tree shape  $S$  that does fold into a polycube  $C$  for  $\mathcal{F} = \{\text{grid}: \text{any}^\circ; \text{interior faces}\}$ , but not in weaker models, in particular, not without interior faces.  $C$  consists of 5 unit cubes; is it minimal? Moreover, we characterized tree shapes that fold into a unit cube in the  $\mathcal{F} = \{\text{grid}: \text{any}^\circ\}$  model—can we characterize polyominoes with holes (possibly of area zero) that fold into a unit cube? If a tree shape folds into a unit cube, can it be folded with rigid faces (continuous blooming)?

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