Testing Simple Polygons

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Abstract: We consider the problem of verifying a simple polygon in the plane using "test points". A *test point* is a geometric probe that takes as input a point in Euclidean space, and returns "+" if the point is inside the object being probed or "−" if it is outside. A verification procedure takes as input a description of a target object, including its location and orientation, and it produces a set of test points that are used to verify whether a test object matches the description. We give a procedure for verifying an n-sided, non-degenerate, simple target polygon using $5n$ test points. This testing strategy works even if the test polygon has $n+1$ vertices. We also give algorithms using $O(n)$ test points for simple polygons that may be degenerate and for test polygons that may have up to $n+2$ vertices. All of these algorithms work for polygons with holes. We also give an extension of the basic testing algorithm to d dimensions.

1 Introduction

Geometric probing [CY87, Ski88, LR89, BS91, Kar91] is the subarea of computational geometry that investigates how to identify or verify an object using a measuring device called a *probe*. In this paper we look at the problem of verifying a simple polygon with probes. For verification problems we are given a description of a target object, including its location and orientation, and we must use probes to verify whether a given test object correctly matches the description. We call the procedure that produces the set of verification probes a *testing algorithm.*

One parameter that is used to specify a geometric probing problem is the type of probe used. In this paper we use point probes or *test points.* Point probes measure whether a single point is inside or outside the object being probed. That is, a point probe takes as input a point in Euclidean space, and it returns + (*positive*) if the point is inside the object being probed or − (*negative*) if it is outside. This type of probe was developed independently by Romanik [RS90b, Rom92] and Mitchell [Mit90]. Here we use the model defined by Romanik, which is defined formally in Section 2 below. This model was used by Romanik and Salzberg [RS92] for obtaining results on verifying orthogonal shapes. Note that in this model exact verification cannot be done since we assume infinite precision and we assume that a point probe does not return information about whether it is on the boundary of an object. Therefore, a testing algorithm can only produce probes that verify an object to within a given error bound.

In this paper we give an explicit procedure for verifying a simple n-sided polygon in the plane by using at most $5n$ test points, provided that the target polygon has no three collinear vertices and the test polygon has at most $n+1$ vertices. We also give strategies using $O(n)$ test points for the case in which the target polygon may have collinear vertices and the case in which the test polygon has at most $n+2$ vertices. If the test polygon can have three vertices more than the target

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polygon, then verification with any finite number of test points is impossible (see Section 4). Previous work by Romanik [Rom92] using this model gave a testing scheme producing $7n$ points for polygons with no three collinear vertices.

Section 2 of this paper gives definitions. Section 3 gives results for simple polygons where both the target and test polygon have n vertices. First a result is given for the case where the target polygon has no two collinear edges, and then the result is generalized to handle collinearities. Section 4 generalizes the results of Section 3 by considering testing strategies for polygons that may have one or two more vertices than the target polygon. Section 5 extends the first result from Section 3 to polyhedra in E^d, where no two facets are coplanar. The full version of this paper is available [A*93].

2 Testing With Point Probes

In this section we give definitions that describe our model for verifying or *testing* with point probes. The model is general enough to be applied to any set of geometric objects.

An *object* q is a measurable subset $q \subset E^d$ of d-dimensional Euclidean space; in particular, it is a Borel set. An *object class* is a set Q of objects. Given a *test object* $r \in Q$ and a *target object* $q \in Q$, r is *consistent* with q on some finite set of test points t if it contains the same subset of t as q, i.e. $t \cap r = t \cap q$. The *error* of r, with respect to q, is given by $V(q \Delta r)$, where $V(p)$ denotes the d-dimensional volume of p and $q \Delta r$ denotes the symmetric difference of the sets. Let S denote the set of all finite sets of points in E^d, let I denote the open interval of rationals $(0, 1)$.

Definition. A computable function $T: Q \times I \rightarrow S$ is a *testing algorithm for Q with test set size k* if there exists a constant k dependent only on Q such that for all $\epsilon \in I$ and for all $q \in Q$, there exists a $t \in S$ such that $T(q, \epsilon) = t$ and $|t| \leq k$ and for all $r \in Q$, if r is consistent with q on t, then $V(q \Delta r) \leq \epsilon$. $T(q, \epsilon)$ is called a *test set for q with respect to the class Q*. For each $t_i \in T(q, \epsilon)$, if $t_i \in q$ then t_i is a *positive test point*; otherwise, t_i is a *negative test point.*

Thus given a target object $q \in Q$ and error bound $\epsilon \in I$, T produces a test set for q such that any test object that is consistent with q on this set has error no more than ϵ. If such a T and k exist, then Q is *k-testable.* Note that in general a testing algorithm may produce test sets whose sizes are functions of both ϵ and the complexity of the target object, but the testing algorithms we develop in this paper produce constant size test sets.

3 Testing n-Sided Polygons

In this section we first develop a testing algorithm for simple polygons with test set size $6n$ (Theorem 1) subject to the following two assumptions, and then improve it to achieve size $5n$ (Corollary 2).

Assumption 1 *The target polygon has no two collinear edges.*

Assumption 2 *The target polygon and the test polygon both have n vertices.*

Theorem 1 *The class of simple polygons in E^2 with n vertices and no collinear edges is $6n$-testable.*

Proof: Our goal is to use 6 test points to verify each edge of the target polygon. The 6 points are divided into 3 pairs of positive/negative points, one pair near each end of the edge, and the third pair "somewhere in the middle of the edge" (see Figure 1). More precisely, each pair is represented by the line segment connecting the two points. Assume that these line segments have some length δ_1, and are placed along an edge. (The orientation of the line segments with respect to the edge is unimportant in the current case. Later, when we remove Assumptions 1 and 2, we must choose this orientation carefully.) One segment is placed at a distance at most some length δ_2 from each of the vertices at the ends of the edge. The third pair of points is placed between them, at a position chosen so that we do not create any unnecessary degeneracies. An *unnecessary degeneracy* is three pairs of points on non-collinear edges that can be split by one line. Think of each pair of test points as being a single point on the boundary of the target polygon (i.e., $\delta_1 = 0$). By Assumption 1, we can place these points such that any line in the plane can stab at most three such points, and if it does stab three, these must be points on one of the target polygon's edges. Now as we increase δ_1 from zero, creating pairs of test points, we can ensure, for small enough δ_1's, that any line in the plane will separate at most three test pairs, and if a line does separate three pairs, these pairs are generated by one edge of the target polygon.

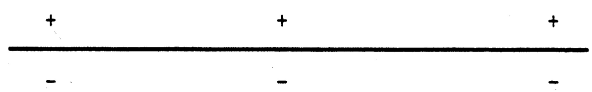

Figure 1: Six points verify an edge of a simple polygon

We use a simple counting argument to show that the edges of a test polygon consistent with the target polygon separate the test pairs in the same circular ordering as does the target polygon (under Assumptions 1 and 2). Any n-sided simple polygon consistent on the test points must separate all $3n$ pairs of points with its n edges. By the placement of the test points, no edge can separate more than two pairs of points except by separating three pairs along an edge. Therefore, in order to separate all $3n$ pairs with n edges, each edge must separate three pairs along some edge of the target polygon.

Finally, we choose parameters δ_1 and δ_2 based on the specific target polygon q and a specific choice of ϵ to guarantee that any n-sided simple polygon r whose edges separate the $3n$ test pairs in the same circular ordering as the target polygon q satisfies $V(q\Delta r) \leq \epsilon$. Clearly, $V(q\Delta r)$ is a continuous function of the parameters δ_i, and decreases to zero as δ_1 and δ_2 approach zero. Thus for any choice of $\epsilon > 0$ one can choose $\delta_1, \delta_2 > 0$ that are small enough to achieve this error bound. □

Note that if a polygon has collinear vertices, but no collinear edges, our testing method of Theorem 1 applies, since we can always place the test pairs so as not to cause any unnecessary degeneracies.

We can reduce the number of test points to $5n$ by "reusing" either one positive or one negative test point at each vertex, depending on whether the interior angle of the polygon is convex or reflex, and making the three test points collinear. However, to do this we must strengthen Assumption 1 as follows:

Assumption 3 *The target polygon has no three collinear vertices.*

Corollary 2 *The class of simple polygons in E^2 with n vertices satisfying Assumption 3 is $5n$-testable.*

Since the proof of Theorem 1 relies only on the general position assumption, it also applies to simple polygons with holes[1].

Corollary 3 *The class of simple polygons with holes in E^2 with n vertices satisfying Assumption 3 is $5n$-testable.*

This proof does not hold for polygons for which two or more edges may be collinear, because a single edge of the test polygon can now separate more than three pairs of test points. As Figure 2 shows, there are polygons that satisfy Assumption 2 but not Assumption 1, where the edges of the test polygon separate different pairs of points than the target. However, we note that these two polygons are very similar by our definition, as the area of the symmetric difference can be made arbitrarily small by shrinking δ_1. Thus, these polygons do not provide a counterexample to the testability of general n-sided polygons.

If we wish to relax our requirement that vertices be in general position, then we can replace Assumption 3 by the following assumption and use $6n$ test points in a strategy similar to the proof of Theorem 1:

[1]The authors would like to thank Jit Bose for suggesting the problem of testing simple polygons with holes.

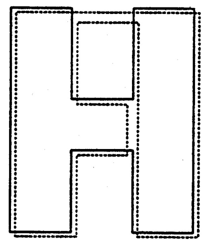

Figure 2: Polygons whose edges separate different pairs of points

Assumption 4 *There exists a bound, $\delta > 0$, such that for each test polygon, the minimum distance between a vertex and an edge of which it is not an endpoint is at least δ.*

Theorem 4 *The class of simple polygons in E^2 satisfying Assumptions 2 and 4 is $6n$-testable.*

Is Assumption 4 necessary? To date we have been unable to identify a target polygon that is not testable with the above scheme, even if Assumption 4 is removed, but we have no proof that the testing scheme is guaranteed to work! Consequently, we eliminate Assumption 4 at the expense of adding more (but still a linear number of) test points. No testing scheme that uses only a linear number of test points can avoid having thin strips by which the target and test polygons may differ, as in Figure 2; that is, no testing scheme can guarantee that clockwise traversals of the test and target polygons separate the test pairs in the same cyclical order. However, our next testing scheme, which builds upon the $6n$-size testing scheme of Theorems 1 and 4, does guarantee that the two polygons have a small symmetric difference.

The revisions to the testing scheme of Theorem 1 focus on collinear edges. Let $e_1 = (v', v)$ and $e_2 = (w, w')$ be collinear edges of the target polygon such that their endpoints are in order v', v, w, w' along the line containing both edges. If no other edge of the target polygon lies along the line segment between v and w, these edges are *consecutive*. If a clockwise traversal of the boundary of the polygon moves from v' to v and from w to w', or from w' to w and from v to v', then these edges are *aligned*. If e_1 and e_2 are consecutive and aligned, then (v, w) will be called a *phantom edge* (depicted by a dashed line in our figures).

Augment the previous testing scheme with three pairs of test points along each such phantom edge, called *phantom edge test pairs* or *phantom pairs*, placed collinear with the test pairs defined by real edges (*real pairs*) and so as to avoid any unnecessary degeneracies. Real pairs have one point of each sign. The phantom pairs along one phantom edge are all of the same sign (all positive or all negative). See Figure 3. If an edge of the test polygon separates pairs of test points generated by two or more collinear edges of the target polygon, then this edge will also erroneously separate the phantom pairs, so in order for the test polygon to be consistent with the target, an additional edge must separate these phantom pairs.

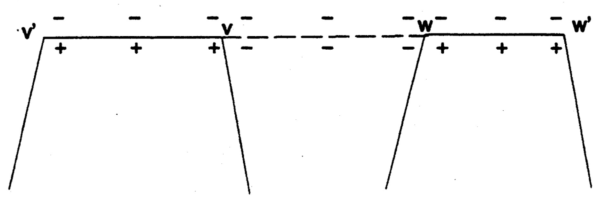

Figure 3: Testing degenerate polygons satisfying Assumption 2.

To formalize this argument, define the *value* of an edge of the test polygon, in a clockwise traversal of the polygon, as the sum of the values of the real and phantom pairs it separates, as follows. If the directed edge separates a real pair, with the positive test point on its right side, the value is +1, whereas if the negative test point is on the right, the value is -1. For a phantom pair, we consider the positive side of the phantom edge generating the pair to be the same as the positive sides of the real edges bounding the phantom edge. The value of a phantom pair is -1 if the pair is separated with the positive side on the right of the separating edge, and +1 if it is on the left. See Figure 4, where the edges of the test polygon are denoted as arrows, and the edges of the target polygon by solid lines.

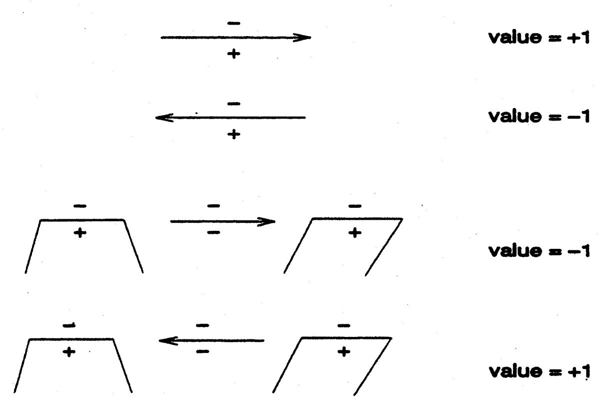

Figure 4: Assignment of value

Lemma 5 *The value of each edge of the test polygon is at most 3.*

Proof: Assume by contradiction that there exists an edge that has value 4 or more. A value of 4 can only be obtained by separating real pairs along consecutive, aligned edges or by separating phantom pairs along aligned phantom edges. However, an edge that separates real pairs on consecutive, aligned edges will also separate the pairs on the phantom edge between these edges, and these pairs will yield the opposite sign. Similarly, an edge that separates pairs from more than one phantom edge will separate real pairs giving the opposite sign. □

Theorem 6 *The class of simple polygons in E^2 with n vertices can be tested with $6n + 6l$ test points, where l is the number of phantom edges (pairs of consecutive, aligned edges) of the target polygon.*

Proof: We use the testing scheme described above. Any test polygon that is consistent with the target polygon on the test points must have value exactly $3n$, since every such polygon must separate phantom pairs an equal number of times in each direction, and real pairs exactly once more with the positive test points on its right. The argument is now reminiscent of that presented in Theorem 1. Since each edge of the test polygon has value at most 3 (Lemma 5), and the number of edges is n, each edge must have value exactly 3. This implies that the three pairs generated by a real or phantom edge of the target polygon are separated by the same edge(s) of the test polygon. As a result, all edges of the test polygon are "close to" edges of the target polygon, and the only place in which the target and test polygons can differ is in the "strips" between the two points of a test pair, as in the example of Figure 2. However, the area of the strips can be made arbitrarily small by decreasing δ_1. Finally, we note that $l \leq n$, so the number of test points remains linear in n. □

The same testing strategy can be used for simple polygons with holes. The edges of a hole are traversed in a counter-clockwise order, so two consecutive collinear edges of the target polygon, $e_1 = (v', v)$ and $e_2 = (w, w')$, where e_2 is an edge of a hole, are *aligned* if a clockwise traversal of the boundary of the polygon moves from v' to v and a counter-clockwise traversal of the boundaries of the holes moves from w to w', or visa versa. As before, a phantom edge occurs between every pair of consecutive, aligned edges. Also, the value of an edge of the test polygon is determined by making a clockwise traversal of the boundary of the polygon and a counter-clockwise traversal of the boundaries of the holes, with the value of a pair being determined as before. By traversing the holes in counter-clockwise order, the arguments of Theorem 6 remain valid, and the following corollary results.

Corollary 7 *The class of simple polygons with holes in E^2 with n vertices can be tested with $6n + 6l$ test points, where l is the number of phantom edges of the target polygon.*

4 Testing Simple Polygons with More Than n Sides

In this section, we consider the question of whether an accurate linear-size testing scheme can be devised when Assumption 2 is relaxed and the test polygon may have more sides than the target polygon. We will assume that the vertices of the target polygon are in general position (Assumption 3). Without Assumption 4, no finite-sized testing scheme is possible.

Theorem 8 *The class of simple polygons (possibly with holes) having n or n+1 vertices, and satisfying Assumptions 3 and 4, can be tested with* $5n$ *(resp.,* $5(n+1)$*) test points for a target polygon of n (resp.,* $n+1$*) sides.*

Proof: We place the test points as in Corollary 2, choosing δ_1 and δ_2 small enough to ensure that the distance from a vertex to any point on either of the two test-segments near it is less than $\frac{\delta}{2}$. The proof proceeds as in Theorem 1: there are $3n$ pairs to be separated, and no one edge of the test polygon can separate more than 3 pairs. Each test pair must be separated by exactly one edge, since two edges separating a pair would cause one of the test points to be inconsistent, and three or more edges separating a pair would cause some vertex to be within δ of a nonadjacent edge, violating Assumption 4. We say that an edge of the target polygon *corresponds to* an edge of the test polygon if the two edges separate the same three pairs of test points. If the test polygon has $n+1$ edges, then there are three cases to consider:

1. All n edges of the target polygon have corresponding edges in the test polygon, and so one edge $f = (v_1, v_2)$ of the test polygon separates no pairs. Let f connect two edges f_1 and f_2 of the test polygon. If the corresponding edges e_1 and e_2 of the target polygon do not share a vertex, then the two edges of the target polygon incident on e_1 have corresponding test polygon edges f_3 and f_4, both of which must share an endpoint with f_1 by Assumption 4. But f and f_1 also share a vertex, providing a contradiction. If e_1 and e_2 do share a vertex v, then f_1 and f_2 both separate a pair of points near the vertex v and pass at most distance $\frac{\delta}{2}$ from vertex v. Thus either v_1 is within distance δ of edge f_2 or v_2 is within δ of f_1, violating Assumption 4.

2. The test polygon r has $n-1$ edges that separate 3 pairs each, and hence have corresponding edges in the target polygon q. Of the remaining two edges of the test polygon, one edge f_1 separates 1 pair, the other f_2 separates 2 pairs, all of which are separated by the single target edge e. Replace f_1 and f_2 by a single edge f separating all three pairs and adjust the adjacent edges to form a new polygon r'. By Theorem 1 we know that for every value of $\epsilon_1 > 0$ we can choose the parameters δ_1 and δ_2 so that $V(q \Delta r') \leq \epsilon_1$. To show that for any $\epsilon_2 > 0$ we can choose the parameters small enough so that $V(r \Delta r') \leq \epsilon_2$, we observe first that as δ_1 is decreased to 0, each test pair becomes a single point on the boundary of the polygon, and $V(r \Delta r') = 0$. As the test points within each pair are moved continuously apart from one another, the maximum potential difference in slope between f_2 and f grows continuously. The limits on the slope of f_2 (determined by δ_1) and on the placement of the vertex v_f shared by f_2 and f_1, (dictated by Assumption 4) prevent f_1 from forming a long needle with either of its adjacent edges when δ_1 is increased from zero, and so the area $V(r \Delta r')$ also grows continuously. Choosing $\epsilon_1 = \epsilon_2 = \epsilon/2$ and the parameters δ_i accordingly, completes the argument.

3. There are $n-2$ edges of the test polygon that separate 3 pairs each, and hence have corresponding edges in the target polygon. Each of the remaining three edges of the test polygon f_1, f_2, and f_3 separates 2 pairs. Since each of the edges f_i separates two pairs of points, it must be that the three edges separate six pairs that are separated by two edges e_1 and e_2 of the target polygon that share a vertex v. Specifically, f_1 (f_2) separates two pairs of test points generated by edge e_1 (e_2) – one middle pair and one pair near the vertex other than v. f_3 must separate the two test pairs near the vertex v. Any other (non-equivalent) configuration would result in the test polygon being non-simple, or a violation of Assumption 4. However, by our placement of test points, no line can separate the two pairs of test points near one vertex, so this case is impossible. □

See [A*93] for the proof to:

Theorem 9 *The class of simple polygons in* E^2 *satisfying Assumptions 1 and 4 with* n, $n+1$, *or* $n+2$ *vertices can be tested with* $7n$ *(resp.,* $7(n+1)$, $7(n+2)$*) test points for a target polygon of n (resp.,* $n+1$, $n+2$*) sides.*

It is tempting to think that by possibly adding some constant number of test pairs per edge we could test polygons with $n+3$ vertices. However, this is false, even with Assumption 4. A test polygon with three more edges than the target polygon can fool any finite-sized testing scheme: create a test polygon by creating a long "needle" somewhere in the middle of one of the externally visible edges of the target polygon, avoiding all test points; this needle replaces one edge with four edges; the area of the symmetric difference between the two polygons can be made arbitrarily large by increasing the length of the needle.

5 Higher Dimensions/Extensions

Consider now the problem of testing polyhedra in E^d. We can extend our two-dimensional result as follows:

Theorem 10 *The class of polyhedra in* E^d *having n facets, no two of which are coplanar, can be tested with*

$2(d+1)n$ test points.

Proof: (Sketch) As in two dimensions, we use pairs of test points that we distribute on facets of the target polyhedron. In each pair of test points, the separation between the points is δ_1, which is small. One point in each pair is a positive point (just inside the target), and one is a negative point (just outside the target).

We distribute $d+1$ pairs on each facet of the target, taking care to make certain that no set of $d+1$ pairs can be separated by a hyperplane, *except* in the case that the $d+1$ pairs are all on a common facet. (That is, we create no unnecessary degeneracies – in general, any d pairs can be separated by a hyperplane, but no $d+1$ pairs on different facets can be separated by a single hyperplane.)

As in the two-dimensional case, any n-faceted test polyhedron that is consistent with the test points must be efficient in separating the $(d+1)n$ test pairs; namely, each facet of the test polyhedron must separate $(d+1)$ test pairs. This follows from the fact that each facet can separate no more than $d+1$ pairs. But this implies that each facet of the test polyhedron must be nearly coincident with some facet of the target polyhedron, separating exactly those test pairs that correspond to the target facet.

We claim that this implies that the volume of the symmetric difference between the test and target polyhedra can be made arbitrarily small by choosing δ_1 sufficiently small. To see this, consider a box B that is large enough to contain the target polyhedron and all the test points. Consider one facet, f, of the target, and let $W_f = B \cap \bigcup_{h \in C_f} h$, where C_f denotes the set of all hyperplanes that separate the $(d+1)$ test pairs of f. As δ_1 goes to 0, the volume of W_f goes to 0; thus, we can, for any given $\epsilon > 0$, pick δ_1 so that the volume of W_f is less than ϵ/n, for every facet f. But the volume of the symmetric difference between the target and test polyhedra is bounded above by $\sum_f vol(W_f)$, so we can pick δ_1 so that this volume is less than ϵ. □

Remarks: (1). Note that the theorem does not assume that the target polyhedron was simply connected — e.g., it may have holes.

(2). The nondegeneracy assumption in the above theorem can likely be removed, if we add some additional test pairs as we did in the two-dimensional case.

Another extension of our results is to the case of curved boundaries. If the target is assumed to have a piecewise-algebraic boundary with constant degree k, then the argument in the above theorem generalizes: Place enough test pairs $(d+k)$ in *general position* on each curved facet so that the only way that a test object can separate all pairs is for it to nearly coincide with the target facet.

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The views, opinions, and/or findings contained in this report are those of the authors and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

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