# A geometric proof of the combinatorial bounds for the number of optimal solutions for the 2-center Euclidean problem

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**Abstract**

**We provide lower and upper bounds for $\gamma(n)$, the number of optimal solutions for the two-center problem: "Given a set $S$ of $n$ points in the real plane, find two closed discs whose union contains all of the points such that the radius of the larger disc is minimized."**

**We present two different geometric proofs of linear upper bounds for $\gamma(n)$. The demonstrated upper bound is exact up to a multiplicative constant and for each $n$ we show a set $S$ of points that allows $n$ optimal solutions. The main result of the paper show the mathing upper and lower bounds for the two-center problem, i.e., we show that $\gamma(n) = n$.**

## 1 Introduction

The two-center problem, "Given a set $S$ of $n$ points in the real plane, find two closed discs whose union contains all of the points and such that the radius of the larger disc is minimized", is an important instance of an intensively studied $k$-center problem where the objective is to find $k$ closed discs that cover $S$ and minimize the maximum radius.

The research interest in $k$-center problems stems both from its practical importance for minimax location problems, and from its stimulating impact on the development of optimization algorithms. It is known that if $k$ is a part of the input, the $k$-center problem is $NP$-complete; see [MS81]. On the other hand, for small values of $k$ there exist efficient algorithms. Particularly impressive is a linear time algorithm for the *one*-center problem presented by Megiddo [Meg84]. For the two-center problem, the best known deterministic algorithm with $O(n^2 \log n)$ complexity is given in [JK94]. Other algorithms that differ only by the polylog factor are given by Agarwal and Sharir [AS91], and by Katz and Sharir [KS93], and a randomized (faster) algorithm presented by Eppstein [Epp91, Epp92]. An important contribution for the decision version of the two-center problem, utilized in the parametric-search approach of [AS91], has been made by Hershberger and Suri [HS91] and later improved by Hershberger.

The present paper is concerned with the combinatorial complexity of the two-center problem. It is well-known, and not difficult to see, that there is a unique optimal solution to the one-center problem. As we demonstrate in this paper, the combinatorics of the two-center problem is very different. We will show that $\gamma(n)$, the number of optimal solutions for the two-center problem, is not bigger than $n$, and that for some sets it is exactly equal to $n$.

The next section provides a precise definition of $\gamma(n)$ and basic geometric properties of the two-center problem. Then we show two completely different geometric proofs for the upper bounds on $\gamma(n)$. The first approach gives a $8n$ upper bound. The other technique results in a sharper $n$, and as it turns out, optimal bound. Section 4 gives an example of a set $S$ that admits $n$ optimal solutions. Finally, in a discussion we present some related open questions.

## 2 Geometric Preliminaries

Let $S$ be a set of $n$ points. We say that a pair $(D^*, \bar{D}^*)$ of two closed discs is optimal for $S$ if all the points in $S$ are contained in the union $D^* \cup \bar{D}^*$, and if $D^*$ is the non-smaller of these two discs and has radius $r^*$ minimal over all pairs of discs whose union contains $S$. $D^*$ will be called ***an optimal disc.*** $\bar{D}^*$ which contains $p \in S \setminus D^*$ will be called a ***complementary*** disc (with respect to $D^*$). Clearly, any optimal pair of discs is a solution to the two-center problem.

There are several simple and well-known properties of the two optimal circles that will be useful through the rest of the paper. We will list them below.

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**Fact 1** Each disc in the optimal pair is determined either by a pair of points in $S$ or by some three of them. That is, the radius of a disc is determined by half the distance between some two points, or is equal to the radius of the circumscribing circle for some three points in $S$.

Note that, since we do not assume general position, the points determining the optimal discs are not unique. However, among the points determining the optimal discs, there are points with special properties. We will formulate these for the case when both discs are defined by three points. The same will hold true for discs defined by two points.

**Fact 2** There exist determining points $p_1, p_2, p_3$ for $D^*$ that do not belong to the interior of $\bar{D}^*$.

**Fact 3** There exist points $p_1, p_2, p_3$ on the circle of the optimal disc such that the triangle $\Delta p_1 p_2 p_3$ is acute. In other words, there are such $p_1, p_2, p_3$ that the center of their circumscribing circle is in the interior of $\Delta p_1 p_2 p_3$.

The above facts follow from an observation that otherwise the optimal disc, contrary to its optimality, could be made smaller. Hereafter, by the ***determining points*** for $D^*$ we will mean points with the above properties. Note that one (or two) of the points determining $D^*$ (the optimal disc) can lie on the boundary of the complementary disc $\bar{D}^*$ but this point does not determine $\bar{D}^*$ in the above sense.

For a given set $S$ we define $\gamma(S)$ to be the number of pairs corresponding to the optimal disc and its complementary disc. Formally,

DEFINITION **2.1** $\gamma(S) = card\{(D, \bar{D}) :$ *D is an optimal disc,* $\bar{D}$ *is its minimal radius complementary disc, for* $S\}$

Note here, that according to the above definition, $\gamma(S) = 4$ for a set $S$ of the vertices of a square; each pair of non-diagonal points determines an optimal disc. Due to the symmetry, complementary discs have the same radius as the optimal discs.

Finally, we have

DEFINITION **2.2** $\gamma(n) = \max_{S \text{ of size } n} \gamma(S)$.

# 3 Upper bounds

In this section we will demonstrate two different approaches for the upper bounds. The first method shows that $\gamma(n) \leq 8n$. This bound is not optimal (the stronger bound will be demonstrated in the second subsection). Yet the proof is slightly simpler than for the stronger one, while it demonstrates basic geometric properties that will be used in the proof of the optimal bound.

## 3.1 Upper bound - first approach

This method is based on counting the number of optimal discs that can pass through determining points $p$ that are closest to the complementary disc. By a packing type of an argument, we will show that at most eight optimal discs can be determined by $p$.

DEFINITION **3.1** *Let $l$ be a line passing through the centers of optimal and the corresponding complementary discs. The distance of a point $p$ on one of these circles to the other circle is defined as the distance of the projection of $p$ on $l$ to the center of this disc.*

Assume that $p$ determines at least three optimal discs $D_1, D_2, D_3$, centered at $O_i$ and determined by $r_i$, and $l_i$, $i = 1, 2, 3$, in addition to $p$. $p$ is assumed to be the closest of the determining points to the minimal complementary disc. The determining points are labelled in such a way that $r_i$ precedes directly $l_i$ on $D_i$ in the counterclockwise order; see figure below. Additionally, denotation is such that $D_2$ is between $D_1$, and $D_3$ when $D_1$ is rotated in the counterclockwise direction about $p$. If any of $D_i$ is determined by two points than we assume that $l_i = r_i$.

We have the following observations

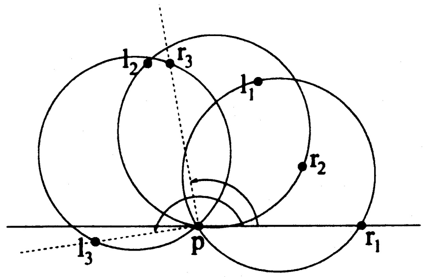
LEMMA **3.1** *$r_1$ and $l_3$ do not belong to $D_2$.*

PROOF: Follows from the counterclockwise organization of $D_1$, $D_2$ and $D_3$. □

LEMMA **3.2** *The angle $r_1 p l_3$ is larger than $\pi$. Angles are measured in the counterclockwise direction.*

PROOF:

The diagonal of $D_2$ passing through $p$ is between $pr_1$ and $pl_3$. Assume that the angle $r_1pl_3$ is not larger than $\pi$. Then $r_1l_3$ intersect $D_2$; it intersects the diagonal of $D_2$. Since $\bar{D}_2$ contains $r_1$, and $l_3$ and does not contain $p$, the other end of the diagonal of $D_2$ beginning at $p$ is contained in $\bar{D}_2$.. Hence, $D_2$ is determined by three points (cannot be determined by two since the other end of the diagonal is inside $\bar{D}_2$), that is $r_2 \neq l_2$ and $r_2$ and $l_2$ lie on the opposite semicircles determined by the diagonal of $D_2$ passing by $p$. But one of these points is closer to $\bar{D}_2$ than $p$. A contradiction. □




LEMMA **3.3** ***The angle $r_1pr_3$ is larger than $\pi/2$.***

**PROOF:** By Lemma 3.2, angle $r_1pl_3$ is greater than $\pi$. Since $p, r_3, l_3$ determine optimal $D_3$, the triangle determined by these points is acute, and in particular the angle $r_3pl_3$ is smaller than $\pi/2$. Hence, $r_1pr_3$ is larger than $\pi/2$. □

THEOREM **3.1** $\gamma(n) \leq 8n$.

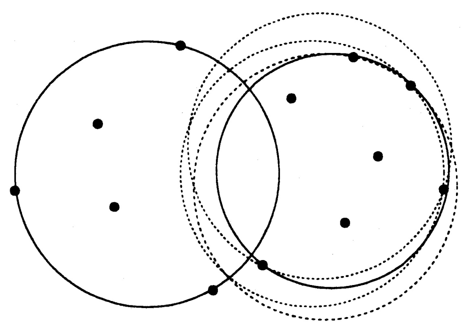
**PROOF:** By Lemma 3.3, the angle $r_ipr_{i+2}$ for every other pair of discs is at least $\pi/2$. Therefore, there are at most eight optimal discs attached to $p$. The bound follows. □

Remark: Althought each optimal disc is determined by at least two points we cannot divide the above $8n$ bound by two; only the closest determining point for each optimal disc is considered in the above arguments.

## 3.2 A sharper bound

The bound for $\gamma(n)$ presented in this subsection is obtained using a different counting strategy. We will count again the number of optimal discs that can be determined by a point $p$ with additional properties defined below. The strategy is based on associating each point determining an optimal disc with an arc of the optimal circle that is intersected by the corresponding complementary disc. We will show that at most two such arcs attached to $p$ can exist for any $p$ in $S$. (Recall that $p$ determines an optimal disc if it is one of the two or three points that determine this disc.)

The first step of the proof is to show that if some optimal disc for $S$ is not properly intersected by its complementary disc (the discs can be tangent to each other), then $S$ admits at most four optimal discs. In this section by a complementary disc we will mean any disc with the radius equal to the optimal radius and such that it contains all the points in $S \setminus D$. Clearly, a complementary disc does not need to be unique. However, the complementary disc $\bar{D}$ whose center is closest to the center of $D$ is unique and has some useful properties.



LEMMA **3.4** *Let $l$ be a line connecting the centers $O_1$ and $O_2$ of $D$ and its closest complementary disc $\bar{D}$. The farther semicircle of $\bar{D}$ determined by the diagonal perpendicular to $l$ contains at least one point of $S$. If this point $w$ is not the intersection of $l$ with the boundary of $\bar{D}$ then there exists another point $p$ of $S$ on the farther semicircle of $\bar{D}$ that is determined by the diagonal passing through $w$.*

PROOF: Otherwise we could move $\bar{D}$ closer to $D$. A contradiction with the choice of $\bar{D}$ as the closest complementary disc of $D$. □

This lemma is symmetric to the following lemma regarding the optimal disc:

LEMMA **3.5** *Any closed semicircle of the optimal disc contains a point of $S$. Any closed semicircle of the optimal disc which is determined by the diagonal passing through a point $w$ of $S$ on this circle contains another point of $S$.*

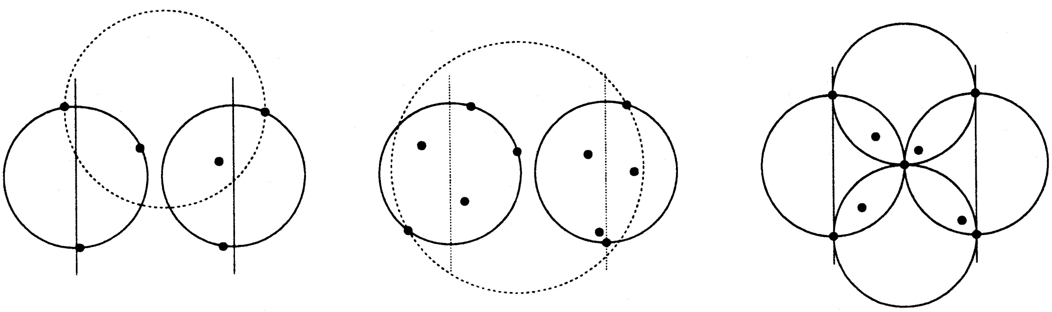
PROOF: Follows from the fact that the points determining an optimal disc are either on a diagonal or they form an acute triangle. □

THEOREM **3.2** *Let a point set $S$ be such that some optimal disc $D$ for $S$ is not properly intersected by its closest complementary discs $\bar{D}$ (it can be tangent). Then there are at most four optimal discs for $S$.*

PROOF: Consider a set $S$ that satisfies the assumptions and the discs do not intersect each other. Let $D$ be an optimal disc and $\bar{D}$ be its complementary disc whose center $O_2$ is closest to the center $O_1$ of $D$ and whose radius is $r$, where $r$ is the optimal radius. Since the discs do not intersect, the length of $O_1O_2$ is at least $2r$. Consider the diagonals of $D$ and $\bar{D}$ perpendicular to $O_1O_2$. There is at least one point of $S$ on each of the farther apart semicircles of $D$ and $\bar{D}$ determined by these diagonals; denote these points by $q$ qnd $\bar{q}$, respectively. For $D$, the existence of $q$ follows from the fact that each semicircle of the optimal circle contains at least one of the determining points, as they are either on a diagonal or form an acute triangle. For $\bar{D}$ it follows from its selection as the complementary disc closest to $D$. (It follows from Lemmas 3.4 and 3.5.) The distance between $q$ and $\bar{q}$ is at least $2r$. By Lemmas 3.4 and 3.5 there exist two additional points on the corresponding semicircles (or a one point on the complementary disc that is on the intersection with the line connecting the centers of the optimal and complementary discs.) For any distribution of these four (or three, respectively) points into an optimal and its complementary disc there is a pair of points whose distance is bigger than $2r$ such that they belong to the same disc. A contradiction; they must be contained in a disc with radius $r$.

If the discs are tangent then the four points implied by Lammas 3.4 and 3.5 determine at most four pairs of the optimal-complementary discs. □

Below, the dashed circles show pairs of points that are farther apart than $2r$.



Based on the above theorem, it is sufficient to consider cases where an optimal disc is intersected by its complementary disc.

We have the following:

LEMMA **3.6** *For any optimal disc $D$, any of its complementary discs $\bar{D}$ can intersect at most one of the arcs on the optimal circle which are determined by the determining points of $D$.*

PROOF: Assume that there are two complementary discs $\bar{D}_1$ and $\bar{D}_2$ such that they intersect two different arcs of $D$ attached to one of the determining points $p$. Then, the set of points of $S$ not in $D$ is also contained in the intersection $\bar{D}_1 \cap \bar{D}_2$. Any disc with radius $r$ and containing this intersection is a complementary disc. But one of them contains $p$;

this contradicts the optimality of $D$. □

The above lemma justifies correctness of the following definition:

DEFINITION **3.2** *Let $D$ be an optimal disc for $S$. The arc of $D$ intersected by a complementary disc $\bar{D}$ is called a neighborhood arc.*

We have a simple lemma:

LEMMA **3.7** *No point of $S \setminus D$ belongs to the convex region determined by the neigborhood arc and the tangents to $D$ at the ends of this arc.*

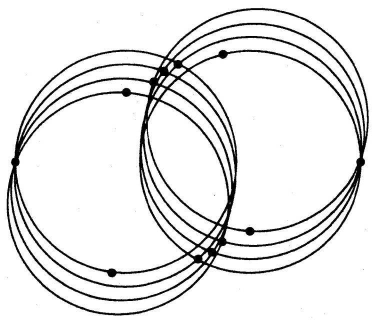
The key element of this proof is captured in the following theorem:

THEOREM **3.3** *For any point $p$ in $S$ there are at most two neighborhood arcs with one of its ends in $p$.*

PROOF: (Sketch) Consider two different optimal discs with $p$ as one of their determining points and such that the neighborhood arcs (with respect to the corresponding complementary discs) have one of their ends in $p$. We will show that the diagonal of the third optimal disc determined by $p$ belongs to the angle between the diagonals containing $p$ of the first and second optimal discs. Then, similarly to the proof of Lemma 3.2 the corresponding neighborhood arc is not connected to $p$. Three different cases arise depending on whether the disc are determined by some two or some three points.

□

Remark: Note that any point $p$ may determine many optimal discs. The above theorem states that at most two of the corresponding complementary discs can intersect their optimal discs at the arc ended at $p$.



As an immediate corollary we obtain

COROLLARY **3.1** $\gamma(n) \leq n$.

PROOF: By Theorem 3.3 each point of $S$ can be associated with at most two pairs of the optimal-complementary discs. On the other hand, each such a pair is associated with at most two points (both ends of the neighborhood arc). Hence, the total number of the optimal pairs is at most $n$. □

# 4 Main result

We will demonstrate an example of a set of $n$ points that admits $n$ optimal discs. This example, together with the results of the previous section, will establish exact bounds for $\gamma(n)$.

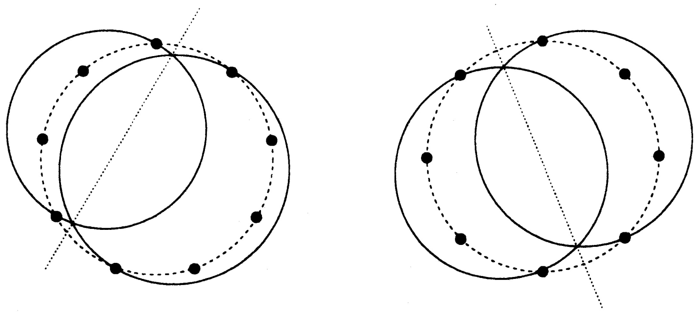
Consider a unit circle and a set $S$ of $n$ equally spaced points on this circle.

THEOREM **4.1** $\gamma(S) = n$.

PROOF: There are two cases two consider. If $n = 2k+1$, then each optimal disc is determined by $k+1$ point of $S$ adjacent on the circle. (The complementary disc contains $k$ points and discs containing more than $k+1$ points have a larger radius.) There are $n$ different such choices of $k$ points. Hence, in this case $\gamma(S) = n$.

If $n = 2k$, then each optimal discs is determined by $k$ adjacent points. There are $n$ choices of such $k$ points. Hence, $\gamma(S) = n$.

□



Remark: Note that for $n = 2k$ in the above proof, the complementary discs have their radius equal to the optimal discs. Therefore, the number of different pairs of the discs is $k$. However, according to our definition, there are $n$ optimal discs determining these pairs.

The above results, together with the upper bound established in Theorem 3.3, gives the following main result of this paper:

THEOREM **4.2** $\gamma(n) = n$.

# 5 Conclusions

We have shown exact upper bounds on $\gamma(n)$, the number of optimal solutions for the two-center problem in the Euclidean plane. Specifically, we have shown that for any set $S$ of $n$ points, the number of optimal solutions is at most $n$, and there exist sets $S$ that admit exactly $n$ solutions. Two different geometric techniques are used for showing the upper bound for $\gamma(n)$. The proof uses elementary geometric properties of circles and optimal discs. Perhaps analytic methods based on a minmax formulation of the two-center problem can give similar bounds.

It is interesting to compare the lower bound $n$ for the two-center problem with the bound for the one-center problem that always admits a unique solution.

Our paper implies the exact bounds of $n$ for another interesting problem of counting the number of optimal pairs such that both the optimal and complementary discs have possibly small radius.

An interesting question is to find bounds for a general $k$-center problem.

# References


[AS91] P. K. Agarwal and M. Sharir. Planar geometric location problems and maintaining the width of a planar set. In *Proc. 2nd ACM-SIAM Sympos. Discrete Algorithms*, pages 449–458, 1991.

[Epp91] D. Eppstein. Dynamic three-dimensional linear programming. In *Proc. 32nd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 488–494, 1991.

[Epp92] D. Eppstein. Dynamic three-dimensional linear programming. *ORSA J. Comput.*, 4:360–368, 1992.

[HS91] J. Hershberger and S. Suri. Finding tailored partitions. *J. Algorithms*, 12:431–463, 1991.

[JK94] J. W. Jaromczyk and M. Kowaluk. An efficient algorithm for the Euclidean two-center problem. In *Proc. 10th Annu. ACM Sympos. Comput. Geom.*, pages 303–311, 1994.

[KS93] M. J. Katz and M. Sharir. An expander-based approach to geometric optimization. In *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, pages 198–207, 1993.

[Meg84] N. Megiddo. Linear programming in linear time when the dimension is fixed. *J. ACM*, 31:114–127, 1984.

[MS81] A. Marchetti-Spaccamela. The $p$-center problem in the plane is NP-complete. In *Proc. 19th Allerton Conf. Commun. Control Comput.*, pages 31–40, 1981.