Motivating Lazy Guards

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1 Introduction

Many variations on the Art Gallery problem have been proposed. A comprehensive list can be found in O'Rourke's book [7] and in Shermer's survey [8]. We propose yet another another variation, that surprisingly has not been studied before. The motivation follows from a common practice with real-life guards.

In real-life, almost any area can be guarded by a single, mobile guard. However, if left to his own devices, a security guard may not patrol as often nor as thoroughly as the employer wishes. To ensure compliance, check-in *stations* that the guard has to physically visit on a regular basis may be installed.

The *lazy guard problem* is thus as follows: Given a polygon, choose a minimum number of stations (points) in the polygon such that a mobile guard that visits all stations will guard the entire polygon. A polygon that can be guarded with k stations is said to be *lazy k guardable.*

2 Lazy guarding a simple polygon

In general we allow the guard to utilize any route between stations, in which case specifying an ordering in which the stations must be visited does not change the problem. If the guard instead follows the shortest path route between stations (the *short lazy guard* problem) then we may distinguish the ordered and unordered versions of the problem. Ordering clearly does not matter for one and two stations. Figure 1 is a polygon with a hole that can be three-station guarded by a short lazy guard (as shown) if the guard must visit the stations in order ABCA but not if the guard visits in order ABACA.

The class of short lazy k guardable polygons is thus a strict superset of lazy k guardable polygons for $k \geq 3$.

Theorem 1 *For simple polygons, ordering stations does not matter and the class of lazy k guardable simple polygons is identical to the class of short lazy k guardable simple polygons.*

Proof:

Suppose a simple polygon is short lazy guardable with k stations visited in some specified order.

Take any point p of the polygon. This point is guarded during a shortest route from some station a to its neighbour b in the specified order. Extend a straight line from p to the route and continue to extend the line until it intersects the far edge of the polygon.

This constructed line divides the polygon into two pieces, one containing a and one containing b. Regardless of the order in which a lazy guard visits the stations, and regardless of

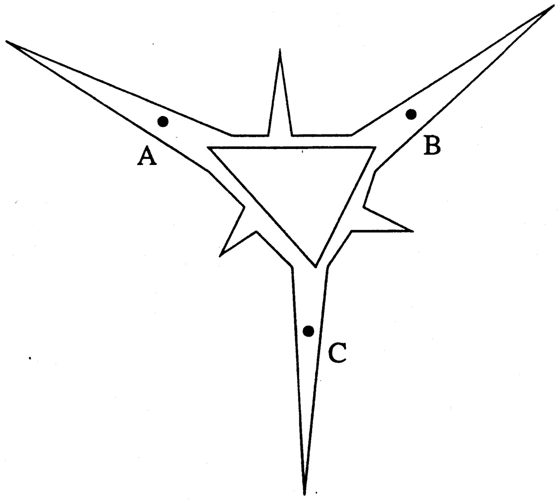

Figure 1: Three-station short lazy guardable with order ABCA

the route taken, the guard must cross this line and thus guard p. ■

For later proofs, it is sometimes convenient to assume that the guard follows the shortest path between stations, and that the stations are ordered so that the guard's route forms a simple (perhaps degenerate) polygon. We call this the *guard's polygon*.

There is no distinction between one guardable polygons and lazy one guardable polygons; both are the class of star polygons, and the guard/station can not always be placed on the boundary of the polygon (figure 2).

For k larger than one, lazy k guardable polygons are a strict superset of k guardable polygons (figure 3).

Some stationary two guardable polygons require the guards to be positioned in the interior of the polygon (refer again to figure 2). However this is not necessary for lazy two guardable simple polygons.

Lemma 2 *Any simple polygon which is lazy two guardable may be guarded with the stations placed on the boundary of the polygon.*

Proof:

From lemma 1, we can restrict ourselves to the case of short lazy two guardable polygons. The guard's polygon is a piecewise linear chain. Extend the end segments of this chain until they intersect the boundary of the polygon. If the stations are moved to these intersection points the shortest route between them is a superset of the original guard's polygon, and thus guards the entire polygon. ■

Theorem 3 *Any simple polygon which is lazy guardable may be guarded with the same number of stations placed on the boundary of the polygon.*

Proof:

Proven already for two guards (lemma 2).

By theorem 1 we consider the guard's polygon, and let s_1 be a station not on the boundary, with neighbours s_0 and s_2. Bisect the exterior angle $\angle s_0 s_1 s_2$ and extend the bisector until it meets the boundary of the polygon at s_1'. Replace station s_1 with a station at s_1'.

The new guard's polygon contains the entire old guard's polygon, and so the polygon is still lazy guarded by the new set of stations. This process can be repeated until all stations are on the boundary of the polygon. ■

A *street* polygon [4] is a simple polygon whose border can be divided into two chains (called left and right respectively) meeting at common points s and t, such that any point on

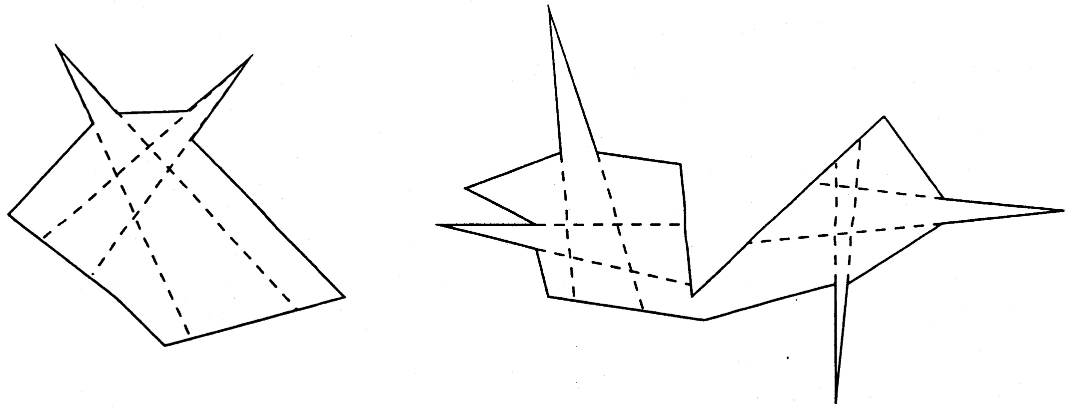

Figure 2: Stationary guardable only with a point in interior of polygon

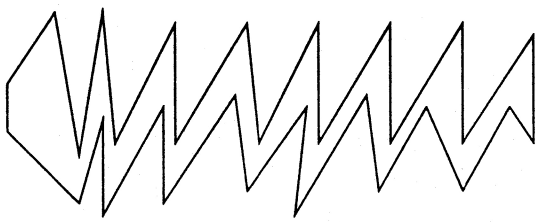

Figure 3: Lazy 2 guardable

the left chain is visible to some point on the right chain, and vice-versa.

We will generalize the notion of a street polygon to more than two vertices. Consider a polygon along with $k \geq 2$ *end points* on the boundary. Each pair of consecutive end points determines a polygonal chain. The polygon along with the end points form a *k-street* polygon if and only if every non-end point can see some point on a different chain (each end point is considered to be on both of the chains meeting at that end point).

Theorem 4 *A simple polygon is lazy k guardable (with $k \geq 2$) if and only if it admits a choice of k end points so as to form a k-street.*

Proof sketch:

Let p be a point on the polygon boundary.

($\Rightarrow$) Move stations to the boundary. Extend a line from p to the guard's polygon and continue until you hit the other side of the polygon. This divides the polygon into two pieces, with stations in both pieces, so the line must connect p to a different chain.

($\Leftarrow$) Place a guard at each of the k end points. Extend a line from p to a point on another chain. This line divides the polygon into two pieces, each containing a station, so p is guarded. ■

We give necessary and sufficient conditions for recognizing a k-street, starting with some notation.

Given two points, s and t, on the boundary of P, we use $C(s,t)$ to denote the subset of the boundary of P traced by a clockwise traversal of P from the point s to t inclusive. Let $\overline{C}(s,t)$ denote the complementary part of the boundary of P. Consider a reflex vertex p in the polygon P. Let p^+ and p^- respectively denote the clockwise and counter-clockwise vertices of P adjacent to p. Extend the ray anchored at p^+ through p and denote the first point of intersection with the boundary of P as $r(p^+)$. Similarly we can obtain $r(p^-)$ by extending the ray from p^- through p. We define the *clockwise component* of the reflex vertex p as the chain $C(p, r(p^+))$. Similarly the *counter-clockwise component* of a reflex vertex p is the chain $C(r(p^-), p)$.

The following characterization is due to Icking and Klein [4].

Lemma 5 *The chain $C(s,t)$ is weakly visible from $\overline{C}(s,t)$ if and only if $C(s,t)$ does not contain a clockwise or counter-clockwise component.*

We can now solve the lazy guard problem.

Algorithm:

1. Compute the clockwise and counter-clockwise components of polygon P. This can be done in linear time [3].
2. Once the components have been identified we can determine a smallest number of points such that each component has at least one point in it. This problem can be described in terms in covering a set of arcs on a circle with a minimum number of points, and is solvable in linear time [5].

Thus we have the following theorem.

Theorem 6 *An optimal placement of stations for lazy guarding a simple polygon can be found in linear time.*

3 Lazy guarding a polygon with holes is NP-complete

Consider the following decision problem.

LG:
INSTANCE: A polygon P (possibly with holes) with vertices at rational coordinates, and an integer k.
QUESTION: Is P lazy k guardable?

Lemma 7 *LG is in NP.*

Proof sketch:
Given a proposed solution consisting of k guard stations, we can verify that the solution is valid in polynomial time.

Extend lines through every pair of visible points (vertices and guard stations) partitioning P into regions consisting of points (where lines intersect), open line segments between intersections, and open regions.

It can be shown that all points in a region are essentially the same. We choose one sample point p from each region and construct its visibility polygon. The point p, and hence its entire region, is guarded if and only if the visibility polygon either contains a station, or partitions P into more than one piece containing a station. ∎

We will reduce a known NP-complete problem to LG. A *vertex cover* of a graph $G = (V, E)$ is a subset of V that contains at least one endpoint of every edge in E.

VCP3:
INSTANCE: G a connected planar graph with degree at most three, and an integer k.
QUESTION: Is there a vertex cover of G of size at most k?

VCP3 is NP-complete [2, 6]. We prove that finding a minimal placement of stationary guards for a polygon with holes is NP-complete. This result is known (a different proof is found in [7]), however, our proof serves as a useful first step for establishing that LG is NP-complete. We describe the stationary guard problem in terms of a polygon cover. Let P be a polygon and let S be a set of points in P (in the interior or on the boundary of P). Let $Vis(S)$ denote the union of the visibility polygons of the points in S.

SG:
INSTANCE: A polygon P (possibly with holes) with vertices at rational coordinates, and an integer k.
QUESTION: Is there a set of points S in P, of cardinality k, such that $Vis(S)$ covers P?

Lemma 8 *SG is NP-complete.*

Proof sketch:
It is a routine matter to verify, in polynomial time, if for a given set of points S in P that $Vis(S)$ covers P. Thus SG is in NP.

We show that VCP3 is polynomially reducible to SG. Let $G = (V, E)$ be a connected planar graph with degree at most three. To create a polygon with holes, P, we obtain a straight line drawing of G such that each vertex is represented by a point, and each edge by a straight line segment. Furthermore, we insist that no vertices in the drawing are on the same horizontal line, and that no two adjacent edges are collinear. That such a drawing can be obtained in polynomial time is well known, see [1]. We now inflate each line segment so that it has some finite width, that is, we map line segments to rectangles. We attach thin spikes to the end of each rectangle. These spikes will enforce a required visibility property. The details follow. See figure 5. Note that to see into the spike a guard must be within the corridor.

It is fairly easy to see that the polygon obtained has all of its vertices at rational coordinates, using a polynomial amount of space. We figuratively call each spiked rectangle corresponding to an edge e of G the *corridor* $c(e)$. Polygon P is the the union of all of the corridors. Because of the spikes, P has the property that a corridor is covered by $Vis(\{s\})$ only if s lies well inside the corridor—the long walls of the rectangle are not visible from part of the spike.

We call the intersection of corridors corresponding to a vertex v, $room(v)$. Suppose that G has a k vertex cover. It is easy to see that by placing a guard in $room(v)$, for every v in the cover of G, we get a stationary guarding of P.

On the other hand, assume that P is guarded by a set of k guards S. Let s be one of the guards in S. If s lies in a set of corridors C then we associate s to a vertex v such that $room(v)$ is in the intersection of the corridors in C.

If every corridor is guarded, then each corridor must have a guard lying in it. Therefore, each edge of G is also covered.

Thus we have shown that P is guarded by k guards if and only if G has a k vertex cover. ■

We proceed by showing that LG is NP-complete.

Theorem 9 *LG is NP-complete.*

Proof:

The reduction is again from VCP3.

We construct a polygon with holes P from a planar graph G, of degree at most three, composed of two components which we call *corridors*, and *ducts*. We obtain the corridor component as in lemma 8. We now augment the corridors with a system of two vertical and $n+1$ horizontal ducts. The vertical ducts are axis parallel rectangles, and the horizontal ducts are hexagons formed by the union of an axis parallel rectangle and two right triangles. The triangles at the end of each horizontal duct are called ***nooks***. See figure 4. Observe that the duct component is lazy guarded if and only if we place a station inside each nook. A path from points x to y avoiding the shaded corridor is shown, illustrating that a station is needed in every corridor to ensure that we lazy guard the corridor component.

The polygon P is constructed so that $2(n+1)$ stations, one per nook, are required to lazy guard the nooks. The remainder of P, that is, the corridors will be lazy guarded by k other stations.

Suppose G can be covered by k vertices. For each vertex v in the cover of G we place a station in $room(v)$. Let S denote these k stations. It follows that $Vis(S)$ covers the corridors, so any tour of these k stations guards the corridor component. By using $2(n+1)$ additional stations, one in each nook, we guard the duct component. Thus P is lazy $2(n+1)+k$ guardable if G is covered by k vertices.

Conversely, suppose P is lazy $2(n+1)+k$ guardable. We have shown that $2(n+1)$ of the stations must be in the nooks, and these stations guard the ducts. For the remaining k stations we use a mapping as in lemma 8. If this mapping leads to a vertex cover of G we are done. Suppose instead that G is not vertex covered. This implies that there is at least one corridor, call it c, that does not contain any stations in it. We argue that we can visit all stations without coming near the centre line of c, thus the spikes of c are not guarded. This contradicts the assumption that P is lazy guarded. Let x and y be arbitrary stations in P. Then there exists points in P that describe a piece-wise linear path (of at most 7 edges) in P that does not see all of corridor c. See figure 4. We can use this method

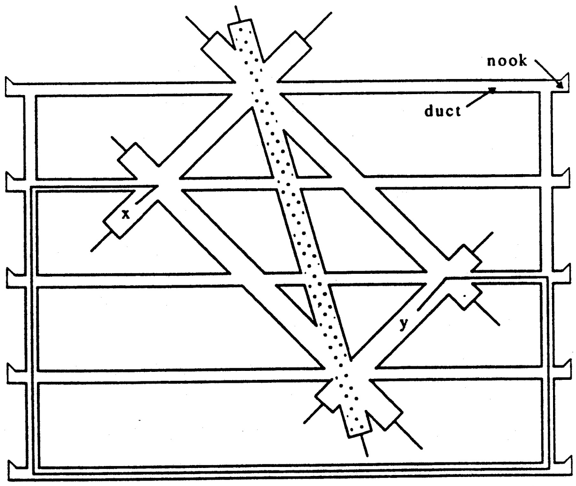

Figure 4: A polygon made up of corridors and ducts.

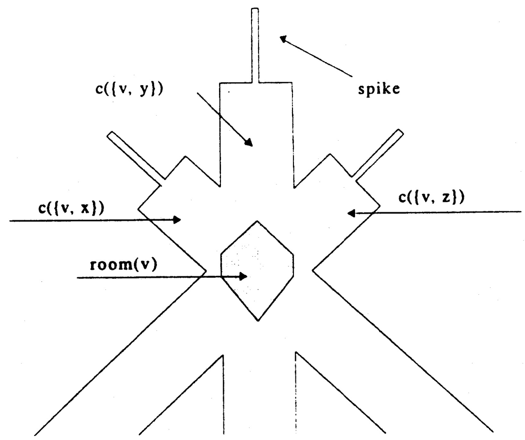

Figure 5: A detail of spiked corridors is shown

to travel between any two stations, and thus obtain a tour of the stations that does not see the spikes of corridor c. This establishes the desired contradiction.

We have shown that P is $2(n+1)+k$ lazy guarded if and only if G is covered by k vertices. Since we have already shown that LG is in NP we conclude that LG is NP-complete. ∎

References

[1] G. Di Battista and P. Eades and R. Tamassia and I. G. Tollis, Algorithms for drawing graphs: an annotated bibliography. Dept. Comput. Sci., Brown Univ., (1993).

[2] M. R. Garey and D. S. Johnson, The rectilinear Steiner tree problem is NP-complete, SIAM J. Appl. Math., 32 (1977) 826–835.

[3] P. J. Heffernan, An optimal algorithm for the two-guard problem, Proc. 9th Annu. ACM Sympos. Comput. Geom. (1993) 348–358.

[4] C. Icking and R. Klein, The two guards problem, Internat. J. Comput. Geom. Appl., Vol. 2, No. 3 (1992) 257–285.

[5] C. C. Lee and D. T. Lee, On a circle-cover minimization problem, Inform. Process. Lett., 18 (1984) 109–115.

[6] D. Lichtenstein, Planar formulae and their uses, SIAM J. Comput. Vol. 11, No. 2, (1982) 329–343.

[7] J. O'Rourke, Art Gallery Theorems and Algorithms, Oxford University Press, New York, NY (1987).

[8] T. C. Shermer, Recent Results in Art Galleries, Proc. IEEE, Vol. 80, No. 9, (1992) 1384–1399