Helly Numbers of Polyominoes

Jean Cardinal * Hiro Ito† Matias Korman* Stefan Langerman*

Abstract

We define the Helly number of a polyomino P as the smallest number h such that the h-Helly property holds for the family of symmetric and translated copies of P on the integer grid. We prove the following: (i) the only polyominoes with Helly number 2 are the rectangles, (ii) there does not exist any polyomino with Helly number 3, (iii) there exist polyminoes of Helly number k for any $k \neq 1, 3$.

1 Introduction

Helly's theorem on convex sets is a cornerstone of discrete geometry, with countless corollaries and extensions in both geometry and combinatorics. For instance, Helly-type properties of convex lattice subsets and hypergraphs have been studied since the 70's [3]. On the other hand, the theory of polyominoes, connected subsets of the square lattice $\mathbb{Z}^2$, has been developed since the 50's with the seminal works of Solomon Golomb [5] and the famous recreational mathematician Martin Gardner.

In this paper, we propose a natural definition of the Helly number of a polyomino P by considering families of symmetric and translated copies of P. We show that the only polyominoes with Helly number 2 are rectangles. We prove the surprising fact that there does not exist any polyomino with Helly number 3. Finally, we exhibit polyominoes of Helly number k for any $k \geq 4$. Since there cannot be polyominoes of Helly number 1, this completely characterizes the values of k for which there exist polyominoes with Helly number k.

Definitions

We define a planar graph $G = (\mathbb{Z}^2, E)$ that represents the adjacency relation between grid points. Each vertex (i, j) is connected to its four neighbors $(i, j-1)$, $(i-1, j)$, $(i+1, j)$, and $(i, j+1)$. A subset of $\mathbb{Z}^2$ is *connected* if its induced subgraph in G is connected.

Definition 1 *A* polyomino *is a connected finite subset of* $\mathbb{Z}^2$*.*

*Computer Science Department, Université Libre de Bruxelles (ULB), Belgium, {jcardin, mkormanc, stefan.langerman}@ulb.ac.be

†School of Informatics, Kyoto University, Japan, itohiro@lab2.kuis.kyoto-u.ac.jp

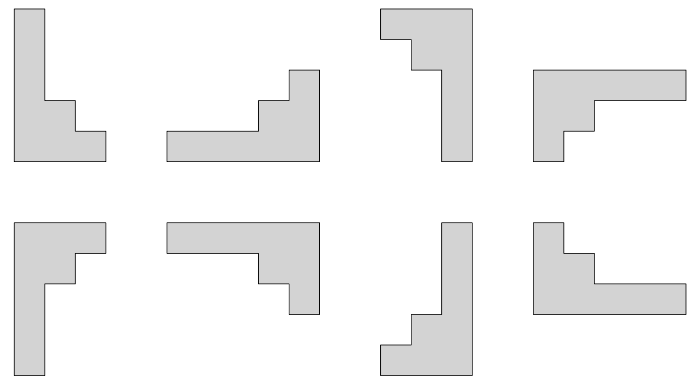

Figure 1: Eight possible symmetries of a polyomino.

We often identify the point $(i, j) \in \mathbb{Z}^2$ with the unit square $[i, i+1] \times [j, j+1] \subset \mathbb{R}^2$. With this transformation a polyomino becomes an orthogonal polygon whose edges are on the unit grid. A *copy* of a polyomino P is the image of P by the composition of an integer translation with one of the eight symmetries of the square (that is, a mirror image and/or a 90, 180, or 270-degree rotation of P). Figure 1 shows an example of a polyomino and its eight symmetries. The cardinality of a polyomino will be denoted by $|P|$ (and will be referred as the *size* of P).

Definition 2 *For any* $k \in \mathbb{N}$ *a polyomino* P *is called* k*-Helly [7] if, for any finite family* $\mathcal{A}$ *of copies of* P *in which* $A_1 \cap \ldots \cap A_k \neq \emptyset$ *for any* $A_1, \ldots, A_k \in \mathcal{A}$*, we have* $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$*. The* Helly number $\mathcal{H}(P)$ *of a polyomino* P *is the smallest* $k \in \mathbb{N}$ *such that* P *is* k*-Helly.*

By definition, any polyomino P that is k-Helly will also be k'-Helly for any $k' \geq k$.

Previous work

A *convex lattice set* in $\mathbb{Z}^d$ is the intersection of a convex set in $\mathbb{R}^d$ with the integer grid $\mathbb{Z}^d$. In 1973, Doignon proved that any family of convex lattice sets in $\mathbb{Z}^d$ is 2^d-Helly [3]. A matching lower bound is obtained by considering all subsets of size $2^d - 1$ of $\{0, 1\}^d$. In our context, this implies that any convex polyomino (i.e. a polyomino that is the intersection a convex set in $\mathbb{R}^2$ with $\mathbb{Z}^2$) is 4-Helly. Note that this is different from the term *convex polyomino*, which usually refers to polyominoes that are simultaneously row and column convex.

Fractional Helly numbers of convex lattice subsets are studied by Bárány and Matousek [1]. Recently, Golumbic, Lipshteyn, and Stern showed that 1-bend paths on a grid have Helly number 3 [6]. We note the environment considered is slightly different, since they considered that two paths have nonempty intersection whenever they share an edge.

2 Helly Number up to 4

In this Section we study polyominoes of small Helly number. Since we are considering finite polyominoes, it is easy to see that no polyomino can have Helly number 1. Thus, we first look for polyominoes with Helly number two.

Definition 3 *A* rectangle *in* $\mathbb{Z}^2$ *is the cartesian product of two intervals in* $\mathbb{Z}$*. The* bounding box *of a polyomino* P *is the smallest rectangle in* $\mathbb{Z}^2$ *that contains* P*.*

It is easy to see that rectangles have Helly number 2. We show that the converse also holds.

Theorem 1 *A polyomino has Helly number 2 if and only if it is a rectangle.*

In the following we give a slightly stronger result; we will show that the only polyominoes that satisfy the 3-Helly property are rectangles.

Definition 4 *A polyomino* P *has the* small empty quadrant *structure if for some copy* P' *of* P*, there exist values* $x_1, y_1 \in \mathbb{Z}$ *such that the intersection of* P' *with the* 2×2 *rectangle* $[x_1, x_1+1] \times [y_1-1, y_1]$ *has cardinality* ≥ 3*, and* P' *contains no point in* $\{(x, y) : x \geq x_1, y > y_1\}$ *(see Figure 2 (a)).*

Definition 5 *A polyomino* P *has the* big empty quadrant *structure if for some copy* P' *of* P*, there exist values* $x_1, y_1, x_2, y_2 \in \mathbb{Z}$*,* $y_1 < y_2$*,* $x_1 < x_2$ *such that* $\{(x_1, y_2), (x_1, y_1), (x_2, y_1)\} \subset P'$ *and* P' *contains no point in the upper right quadrant* $\{(x, y) : x > x_1, y > y_1\}$ *(see Figure 2 (b)).*

Given a rectangle $[x_0, x_1] \times [y_0, y_1]$, its *height* is $y_1 - y_0 + 1$. Analogously, its *width* is $x_1 - x_0 + 1$. The height and width of a polyomino P are equal to the height and width of the smallest enclosing rectangle of P, respectively.

Lemma 2 *Every polyomino* P *whose height and width are 2 or more either has the small empty quadrant or the big empty quadrant structure.*

Proof. Observe that if P has either height or width exactly 1 it must be a rectangle. Hence, in particular, this Lemma shows that any polyomino (other than some

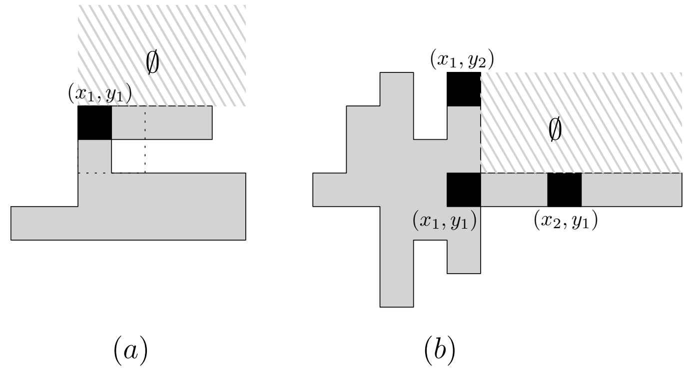

Figure 2: Illustration of Lemma 2. In order for a polyomino P (of height at least 2) to not have the small empty quadrant structure (case (a)), P cannot have two consecutive points on its upper boundary. If this occurs, we can find a large empty quadrant (case (b)). The coordinates x_1, x_2, y_1, y_2 that generate the big or small empty quadrant are shown in black.

rectangles), has one of the two structures. Let (x_0, y_0) be the point of P highest x-coordinate along the upper boundary of its bounding box.

We first show that if $(x_0, y_0-1) \notin P$, then there exists $i \in \mathbb{N}$ such that $(x_0-i+1, y_0), (x_0-i, y_0), (x_0-i, y_0-1) \in P$. Proof is as follows: by definition of (x_0, y_0), we have that $(x_0 + 1, y_0) \notin P$, and $(x_0, y_0 + 1) \notin P$. If we suppose that $(x_0, y_0 - 1) \notin P$, then, in order for P to be connected, we must have $(x_0 - 1, y_0) \in P$. By applying the same argument iteratively on this new point, we must have that eventually there exists an i such that both $(x_0 - i - 1, y_0) \in P$ and $(x_0 - i - 1, y_0 - 1) \in P$, otherwise P is a rectangle of height 1.

Therefore, if $(x_0, y_0 - 1) \notin P$, P has the small empty quadrant structure. Now assume otherwise and let j be the smallest integer such that $(x_0, y_0 - j) \in P$ and $(x_0, y_0 - j - 1) \notin P$. If the quadrant $\{(x, y) : x > x_0, y \geq y_0 - j\}$ contains no point of P, then, by the same argument as in the above claim, there must be a point of P immediately left of the column x_0 between y_0 and $y_0 - j$. In other words, there must be an integer $j' \in [0, j-1]$ such that $|P \cap ([x_0-1, x_0] \times [y_0-j'-1, y_0-j'])| \geq 3$, and again P has the small empty quadrant structure.

Finally, if the quadrant $\{(x, y) : x > x_0, y \geq y_0 - j\}$ is not empty, let (x', y') be the highest point in that quadrant (pick one arbitrarily if many exist). In that case, the three points $(x_0, y_0), (x_0, y'), (x', y')$ form a big empty quadrant structure. □

Lemma 3 *If a polyomino* P *has the big empty quadrant structure, then* $\mathcal{H}(P) \geq 4$*.*

Proof. We construct an arrangement of four copies of P such that every subset of three copies have a common point, but there is no point common to all four copies. We denote these copies by P_i, with $i = 1, \ldots, 4$.

Consider the three points (x_1, y_2), (x_1, y_1), and (x_2, y_1) given by the big empty quadrant structure in P. We construct the copies P_i by flipping P around the x and/or y axis so that those three points map to all possible triples of points in the set $\{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\}$. Since $(x_2, y_2) \notin P$, each of the four points is missing from exactly one copy P_i, but belongs to the other three.

Now we observe that the empty quadrants of the four copies P_i cover $\mathbb{Z}^2$. Hence for any $(x, y) \in \mathbb{Z}^2$, there exists at least one $i \in \{1, 2, 3, 4\}$ such that $(x, y) \notin P_i$. Therefore, the four copies have no common intersection point. □

We now consider polyominoes that have the small empty quadrant structure. We will use the following observation.

Observation 1 *For any polyomino P that is not a rectangle, there exists a 2×2 rectangle R such that $|P \cap R| = 3$.*

Lemma 4 *If a polyomino P has the small empty quadrant structure and is not a rectangle, then $\mathcal{H}(P) \geq 4$.*

Proof. We construct an arrangement of at most 8 copies of P such that every subset of three copies have a common point, but there is no point common to all copies. Let (x_1, y_1) be the point given by the small empty quadrant structure, and P' the corresponding copy of P.

We first consider the case in which the intersection L of P' with the 2×2 rectangle $[x_1, x_1 + 1] \times [y_1 - 1, y_1]$ has cardinality exactly 3. In that case, we can use a similar construction as in Lemma 3, with four copies of P; we define the copies P_i for $i = 1, 2, 3, 4$ as the four rotations of P that map the bounding box of L to the same 2×2 rectangle. Those four points are the respective intersection points of all four possible triples. Similar to the previous case, the four empty quadrants cover all the other points of $\mathbb{Z}^2$, hence there cannot be a common intersection point.

It remains to consider the case in which the intersection L has size 4. In this situation we use the same construction, but complete it with four more copies. From Observation 1 and the fact that P is not a rectangle, we know that there exists a 2×2 rectangle R such that $|P' \cap R| = 3$. We add four additional copies P_i, with $i = 5, 6, 7, 8$, that are the four rotations of a translated copy of P' mapping R to the bounding box of L. Each of the four points of this rectangle belongs to copies P_1, P_2, P_3, P_4 (since $|L| = 4$), and to exactly three of the four copies P_5, P_6, P_7, P_8 (since $|P' \cap R| = 3$). Hence every triple of copies intersects. However, from the previous construction, there still exists no point common to all 8 copies. This construction does not work for rectangles, since Observation 1 does not hold in that case. □

Corollary 5 *There is no polyomino of Helly number 3.*

Combining this result with the upper bound of [3], we can compute the Helly number of any convex polyomino:

Corollary 6 *Let P be a polyomino that is the intersection a convex set in $\mathbb{R}^2$ with $\mathbb{Z}^2$. If P is a rectangle then $\mathcal{H}(P) = 2$. Otherwise $\mathcal{H}(P) = 4$.*

3 Hypergraph Generalization

In this section we study some interesting properties of polyominoes of Helly number k. Since these results hold for subsets of a discrete set of points, we state these results in a more general fashion. Instead of copies of a given polyomino we can consider the same definitions for families of subsets of $\mathbb{Z}^2$. Using this idea, one can extend the Helly property to hypergraphs.

Definition 6 *A hypergraph $G = (V, \mathcal{E})$ is k-Helly if for any $\mathcal{W} \subseteq \mathcal{E}$ such that $e_1 \cap \ldots \cap e_k \neq \emptyset$ for all $e_1, \ldots, e_k \in \mathcal{W}$, we have $\cap_{e \in \mathcal{W}} e \neq \emptyset$. The* Helly number *$\mathcal{H}(G)$ of a hypergraph G is the smallest value k such that G is k-Helly.*

Helly numbers of hypergraphs have been deeply studied (see for example the survey of Dourado, Protti, and Szwarcfiter [4]). Observe that the above definition is a generalization of the previous definition for the polyomino case. Indeed, the polyomino formulation is the particular case in which $V = \mathbb{Z}^2$ and $\mathcal{E}$ contains all subsets of points contained in copies of a fixed polyomino P.

Let G be a hypergraph that is not k-Helly. By definition, there exists a subset $\mathcal{W} \subseteq \mathcal{E}$ such that $\cap_{e \in \mathcal{W}} e = \emptyset$ and $e_1 \cap \ldots \cap e_k \neq \emptyset$ for any $e_1, \ldots, e_k \in \mathcal{W}$. Any such family is called a a *k-witness* set of G. For every $V' \subset V$, define the *restriction* of G to V' as $G|_{V'} = (V', \mathcal{E}|_{V'})$, where $\mathcal{E}|_{V'} = \{e \cap V' | e \in \mathcal{E}\}$. With these definitions we can prove an upper bound on the Helly number of any hypergraph:

Theorem 7 *Let $G = (V, \mathcal{E})$ be a hypergraph. If $|e| \leq k$ $\forall e \in \mathcal{E}$, then G is $(k + 1)$-Helly.*

Proof. We will show the result by induction on k. Observe that the claim for $k = 0$ is trivial, hence we focus on the induction step. Assume otherwise: let $\mathcal{W} \subseteq \mathcal{E}$ be a $(k + 1)$-witness set, and e be an edge of maximum size among those of $\mathcal{W}$ (by hypothesis we know that $|e| \leq k$).

Consider the hypergraph $G' = (e, \mathcal{W}|_e \setminus \{e\})$ (that is, we disregard all other vertices except those contained in e). Since $|e| \leq k$, its intersection with any other edge of $\mathcal{W}$ must be of size at most $k-1$. Furthermore, every k-tuple of edges in G' have a common intersection (since every $k+1$ tuple in $\mathcal{W}$ including e had a common intersection). Therefore, by induction G' is k-Helly. In particular all edges in G' have a common intersection, which by construction intersects e and contradicts the witness property. □

Corollary 8 *Any polyomino P satisfies $\mathcal{H}(P) \leq |P|+1$.*

The proof is direct from the fact that the associated hypergraph is $|P|$-uniform. We also note that the bound of Corollary 8 is tight: the polyomino $\{(0,0),(1,0),(0,1)\}$ (commonly referred as *El* [2]) has cardinality 3 and contains the small empty quadrant structure. In particular, by Lemma 4 its Helly number must be at least 4.

In the following we give a few more tools to use when proving that a given hypergraph is k-Helly (or equivalently, that there cannot exist a k-witness).

Lemma 9 *Any k-witness $\mathcal{W}$ of a hypergraph G satisfies $|\mathcal{W}| \geq k+1$ and $|e_1 \cap \ldots \cap e_\ell| \geq k-\ell+1$ for all $e_1, \ldots, e_\ell \in \mathcal{W}$.*

Proof. Observe that the first claim is trivial, since if $\mathcal{W}$ has size k or less it cannot have an empty intersection. The proof of the second claim is by contradiction: assume otherwise and let $e_1, \ldots, e_\ell \in \mathcal{W}$ such that such that $e_1 \cap \ldots \cap e_\ell = \{v_1, \ldots, v_m\}$ for some $m \leq k-\ell$. Since $\cap_{e\in\mathcal{W}} e = \emptyset$, for any $i \leq k-\ell$ there exists $f_i \in \mathcal{W}$ such that $v_i \notin f_i$.

Consider now the intersection of $e_1 \cap \ldots \cap e_\ell \cap f_1 \cap \ldots \cap f_m$: by construction, this set is empty. Moreover, the size of the set $\{e_1, \ldots, e_\ell, f_1, \ldots, f_m\}$ is at most $\ell+m \leq \ell+k-\ell = k$, which contradicts the witness property of $\mathcal{W}$. □

For any hypergraph G and vertex $v \in V$, we define $c_v = \{e \in \mathcal{W}, v \in e\}$ as the edges that contain v. In the following we show that we can ignore vertices that are not heavily covered.

Lemma 10 *Let $\mathcal{W}$ be a k-witness set of G and let $V' = \{v \in V, |c_v| \geq k\}$. The set $\mathcal{W}|_{V'}$ is a k-witness for $G|_{V'}$.*

Proof. Observe that $\cap_{e\in\mathcal{W}} e = \emptyset \Rightarrow \cap_{e\in\mathcal{W}|_{V'}} e = \emptyset$. Hence, it suffices to show that $e_1 \cap \ldots \cap e_k \cap V' \neq \emptyset$, for any $e_1, \ldots, e_k \in \mathcal{W}$,

Let $S = e_1 \cap \ldots \cap e_k$. Observe that, since $\mathcal{W}$ is a witness set, we have $S \neq \emptyset$. Moreover all points of S are covered by at least k hyperedges (since they are contained in $e_1, \ldots, e_k$). Hence we have $S \subseteq V'$. In particular, we obtain $e_1 \cap \ldots \cap e_k = e_1 \cap \ldots \cap e_k \cap V' \neq \emptyset$ which proves the Lemma. □

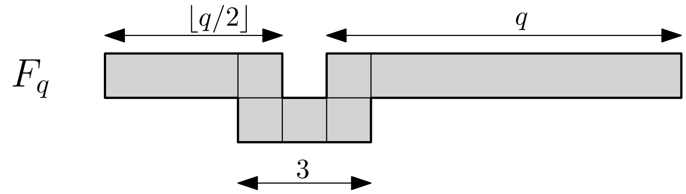

Figure 3: Polyomino F_q.

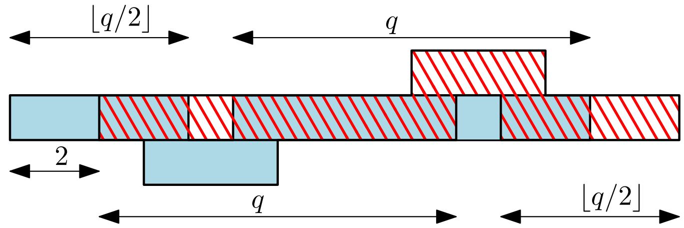

Figure 4: Polyominoes A_0 (solid blue) and B_2 (dashed in red). In the example $q=8$.

Lemma 9 gives a lower bound on the size of a witness set. We use a similar reasoning to find an upper bound as well:

Lemma 11 *Let G be any hypergraph such that $\mathcal{H}(G) = k$. There exists a $(k-1)$-witness set $\mathcal{W} \subseteq \mathcal{E}$ of P such that $|\mathcal{W}| = k$.*

Proof. Let $\mathcal{W}_{\min}$ be the $(k-1)$-witness set of smallest size (pick any arbitrarily if many exist) and let $m = |\mathcal{W}_{\min}|$. By Lemma 9 we have $m \geq k$. If $m = k$ we are done, thus we focus in the $m > k$ case.

By minimality of $\mathcal{W}_{\min}$, there cannot exist a proper subset $\mathcal{W}' \subset \mathcal{W}_{\min}$ such that $\cap_{A\in\mathcal{W}'} A = \emptyset$ (otherwise we would have a witness set of smaller size). In particular, any subset $\{e_1, \ldots, e_k\} \subset \mathcal{W}_{\min}$ must have non-empty intersection. Since G is k-Helly, we have $\cap_{e\in\mathcal{W}_{\min}} e \neq \emptyset$ which contradicts the witness property. □

4 Higher Helly Numbers

In the following we use the above tools to show the existence of polyominoes of Helly number k (for any $k \geq 5$). For any $q \in \mathbb{N}$, we define polyomino F_q is defined as the union of rectangles $[-\lfloor q/2 \rfloor, -1] \times [0,0]$, $[1,q] \times [0,0]$ and $[-1,1] \times [1,1]$. Observe that $|F_q| = \lfloor 3q/2 \rfloor + 3$, see Figure 3.

Lemma 12 *For any $q \geq 4$, we have $\mathcal{H}(F_q) = q+1$.*

Proof. We show the lower bound by constructing a q-witness set $\mathcal{W}$ of F_q. For any $i \leq q$, we define A_i as the copy of F_q translated such that the leftmost point is at position $(i,0)$. Analogously, we define polyomino B_i as the 180-degree rotation of F_q translated so as the leftmost point is at position $(i,0)$

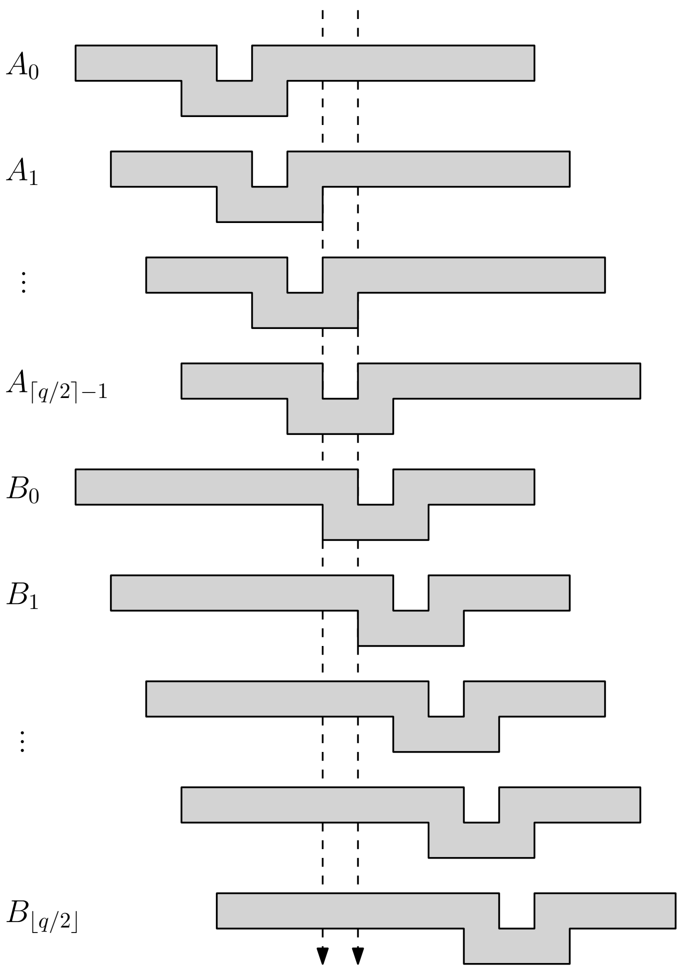

Figure 5: q-Witness set for polyomino F_q (for clarity, each of the copies has been shifted vertically). Observe that, although the intersection of the witness set is empty, any q elements of the set have nonempty intersection. In the figure, we depicted with a vertical strip the point that is contained in all polyominoes except $A_{\lceil q/2 \rceil} - 1$.

(see Figure 4). We define the witness set as $\mathcal{W} = \{A_0, \ldots, A_{\lceil q/2 \rceil - 1}, B_0, B_0, \ldots, B_{\lfloor q/2 \rfloor}\}$. Observe that $|\mathcal{W}| = \lceil q/2 \rceil + \lfloor q/2 \rfloor + 1 = q + 1$ and that the intersection between polyominoes A_i and B_j is in the rectangle $[0, \lfloor 3q/2 \rfloor] \times [0, 0]$ (for any i and j).

More interestingly, for any $0 \leq i \leq \lceil q/2 \rceil - 1$, polyomino A_i does not contain point $(\lfloor q/2 \rfloor + i, 0)$ (and this point is contained in all other polyominoes). The same result holds for polyomino B_i: for any $0 \leq i \leq \lfloor q/2 \rfloor$, point $(q+i, 0)$ is contained in all polyominoes except B_i. In particular, we have $\cap_{C \in \mathcal{W}} C = \emptyset$ and any subset of size q has nonempty intersection (see Figure 5). Hence, $\mathcal{W}$ is a q-witness set of F_q.

In order to finish the proof of the Lemma, we must show that polyomino F_q indeed is $(q+1)$-Helly. Assume that F_q is not $(q+1)$-Helly. Let $\mathcal{W}$ be a $(q+1)$-witness set and let A be the leftmost copy of F_q in $\mathcal{W}$ (pick any arbitrarily if more than one exist). Without loss of generality, we can assume that $A = A_0$. By Lemma 9, there must exist at least $q+1$ other copies A of F_q such that $|A \cap A_0| \geq q$.

First notice that if any two copies of the polyomino do not align their longest segment horizontally, they only have an intersection of size at most 4 with A_0. Moreover, the only case when this intersection has size 4 is if they are two copies flipped across the horizontal axis. In the latter case, any further copy can have an intersection of size at most 3 with at least one of those two copies. Since in either case we obtain a contradiction with Lemma 9 and the fact that $q \geq 4$, we can assume that for any $q+1$-witness set, all copies of $\mathcal{W}$ are aligned horizontally.

Consider now the 3 lower points $(\lfloor q/2 \rfloor - 1, -1), (\lfloor q/2 \rfloor, -1)$ and $(\lfloor q/2 \rfloor + 1, -1)$ of A_0. Since A_0 is the leftmost copy of P and $q \geq 4$ and copies are aligned horizontally, the three points can only be covered by at most two other copies (A_1 and A_2). Therefore we apply Lemma 10 to show that any $(q+1)$-witness set of $\mathbb{Z}^2$ would be a witness set of $\mathbb{Z}^2 \setminus \{(\lfloor q/2 \rfloor - 1, -1), (\lfloor q/2 \rfloor, -1), (\lfloor q/2 \rfloor + 1, -1)\}$. Thus, we focus our attention in the rectangle $[0, \lfloor 3q/2 \rfloor] \times [0, 0]$.

Observe that, since we are considering only this rectangle, the extra copies caused by reflections across the horizontal axis are eliminated because they become the same hyperedge in the restricted hypergraph. Hence, all elements of $\mathcal{W}$ must be of the form A_i or B_j for some $i, j \geq 0$. Also notice that we have $|A_0 \cap A_i| \geq q$ if and only if $i \in \{1, \ldots, \lfloor q/2 \rfloor - 1\}$ (provided that $q \geq 4$). Analogously, if $q \geq 2$ we have $|A_0 \cap B_j| \geq q \Leftrightarrow j \in \{0, \ldots, \lfloor q/2 \rfloor - 1\}$. In particular, the set $\mathcal{W}$ can have at most $2\lfloor q/2 \rfloor$ elements, hence there cannot exist a $(q+1)$-witness set.

□

Theorem 13 *For any $k \in \mathbb{N}$ such that $k \neq 1, 3$, there exists a polyomino P such that $\mathcal{H}(P) = k$.*

5 Conclusion

In this paper we have completely characterized for which values of k there exist polyominoes of Helly number k. An interesting problem is to find a method to compute the Helly number of a given polyomino. Using the results of Section 3, it is not hard to devise an algorithm that runs in exponential time, testing all possible witness sets. Although finding an algorithm that works for general hypergraphs is difficult [4], we wonder whether one can devise an algorithm that runs in polynomial time for any given polyomino P.

Finally note that we defined a copy of P as any image of P with respect to translations and the 8 symmetries of the square. Our results do not hold if we only consider translations (or rotations and translations). Hence, it would be interesting to see how much can the Helly number of a given polyomino change when allowing or forbidding these operations.

References

[1] I. Bárány and J. Matousek. A fractional Helly theorem for convex lattice sets. *Advances in Mathematics*, 174(2):227 – 235, 2003.

[2] J. Beck. *Combinatorial games: Tic–Tac–Toe theory*. Encyclopedia of mathematics and its applications. Cambridge Univ. Press, Cambridge [u.a.], 2008.

[3] J.-P. Doignon. Convexity in crystallographic lattices. *Journal of Geometry*, 3:71–85, 1973.

[4] M. C. Dourado, F. Protti, and J. L. Szwarcfiter. Computational aspects of the helly property: a survey. *Journal of the Brazilian Computer Society*, 12(1):7–33, 2006.

[5] S. W. Golomb. *Polyominoes: Puzzles, Patterns, Problems, and Packings*. Princeton University Press, 2nd edition, 1996.

[6] M. C. Golumbic, M. Lipshteyn, and M. Stern. Edge intersection graphs of single bend paths on a grid. *Networks*, 54(3):130–138, 2009.

[7] J. E. Goodman and J. O'Rourke, editors. *Handbook of discrete and computational geometry*. CRC Press, Inc., Boca Raton, FL, USA, 2004.