An Incremental Algorithm for High Order Maximum Voronoi Diagram Construction

Khuong Vu * Rong Zheng*

Abstract

We propose an incremental approach to compute the order-k maximum Voronoi diagram of disks in the plane. In our approach, we start with an order-k Voronoi diagram of disk centers and iteratively expand disks and update the changes of the diagram until all disks reach their targeted size. When disks expand continuously, the structure of the diagram changes discretely. The algorithm takes $O\left(\left\lceil \frac{r_{\max}-r_{\min}}{d_{\min}} \right\rceil k^2 N \log N\right)$ time complexity, where N, $r_{\max}$ and $r_{\min}$ are respectively the number of disks, the maximum and minimum radii of disks, and $d_{\min}$ is the minimum distance between two disk centers. Our algorithm is amiable to distributed implementation.

1 Introduction

Consider a set of N disks $S = \{\mathcal{D}_1(o_1, r_1), \mathcal{D}_2(o_2, r_2), \ldots, \mathcal{D}_n(o_N, r_N)\}$, where o_i and r_i are respectively the center and radius of disk $\mathcal{D}_i$ $(1 \leq i \leq N)$. We define the distance from a point p to disk $\mathcal{D}_i$ as $d_{\max}(p, \mathcal{D}_i) = d(p, o_i) + r_i$, where $d(\cdot, \cdot)$ is the Euclidean distance between two points. The locus of points closer to $\mathcal{D}_i$ than to $\mathcal{D}_j$, $h(\mathcal{D}_i, \mathcal{D}_j)$, is one of the half-planes determined by the bisector $b(\mathcal{D}_i, \mathcal{D}_j) = \{p | d_{\max}(p, \mathcal{D}_i) = d_{\max}(p, \mathcal{D}_j)\}$, or $b(\mathcal{D}_i, \mathcal{D}_j) = \{p | d(p, o_i) - d(p, o_j) = r_j - r_i\}$. In general, $b(\mathcal{D}_i, \mathcal{D}_j)$ is a hyperbolic curve. Let $\mathcal{V}(\mathcal{D}_i)$ denote the locus of points closer to $\mathcal{D}_i$ than to any other disk in S. Thus, $\mathcal{V}(\mathcal{D}_i) = \bigcap_{i \neq j} h(\mathcal{D}_i, \mathcal{D}_j)$. It has been shown in [4] that if $\mathcal{D}_i$ does not contain any other disks, then $\mathcal{V}(\mathcal{D}_i) \neq \emptyset$. In addition, $\mathcal{V}(\mathcal{D}_i)$'s boundary consists of edges, which are hyperbolic segments, and vertices, which are intersections of adjacent edges. $\mathcal{V}(\mathcal{D}_i)$ is referred to as the Voronoi face associated with disk $\mathcal{D}_i$, and the set $\{\mathcal{V}(\mathcal{D}_i), 1 \leq i \leq n\}$ is referred to as the *maximum Voronoi diagram*, or *max VD* for short, of S. In [4], the authors proposed an algorithm to construct the max VD of n disks in $O(T(N) + N \log N)$, where $T(N)$ is the time of the nearest neighbor query under $d_{\max}$ metric. An example of max VD is shown in Figure 1. As seen in the figure, $\mathcal{V}(\mathcal{D}_8) = \emptyset$ since it contains $\mathcal{D}_6$.

*Department of Computer Science, University of Houston, {khuong.vu, rzheng}@cs.uh.edu

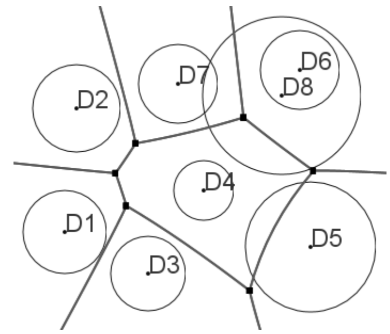


Figure 1: The max Voronoi diagram of 8 disks.

Similar to the high order Voronoi diagram of points first studied by Lee [8], we generalize the concept of max VD such that a Voronoi face is associated with a set of disks, $H \subset S$ for $|H| > 1$. Denote $\mathcal{V}^k(H, S)$, where $|H| = k, H \subset S$, the locus of points closer to *all* disks of H than to *any* disk in $S \setminus H$. We define the order-k max VD of S, $V^k(S)$, as a collection of Voronoi faces corresponding to all subsets H of S $(|H| = k)$, i.e., $V^k(S) = \bigcup_{H \subset S} \mathcal{V}^k(H, S), |H| = k$. We adopt the definition in [3] to formally define the order-k max VD as follows:

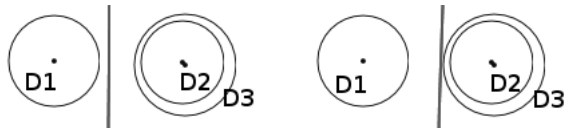
Definition 1 *For i, $j \in S$, let $D(\mathcal{D}_i, \mathcal{D}_j) = \{p | d_{\max}(p, \mathcal{D}_i) < d_{\max}(p, \mathcal{D}_j)\}$. Let $H \subset S, |H| = k$. We define*

$$\mathcal{V}^k(H, S) = \bigcap_{h \in H, i \notin H} D(\mathcal{D}_h, \mathcal{D}_i)$$

the order-k maximum-Voronoi face of a set of disks H with respect to S. The order-k maximum-Voronoi diagram of S is defined as

$$V^k(S) = \bigcup_{H, H' \subset S; H \neq H'; |H| = |H'| = k} \mathcal{V}^k(H, S) \bigcap \mathcal{V}^k(H', S)$$

In [10], Lee's incremental algorithm is applied to construct order-k max VD under the assumption that *no disk contains any other disk.* Accordingly, each Voronoi face of order-$(k-1)$, $\mathcal{V}^{k-1}(H, S)$, is tessellated by the order-1 Voronoi diagram of some disks to create the next order Voronoi faces. It has been shown that only disks associated with edges of $\mathcal{V}^k(H, S)$ need to be considered for tessellation. In general placement of disks, the



(a) order-1 max VD. The curve is the bisector of disks $\mathcal{D}_1$ and $\mathcal{D}_2$. $\mathcal{V}(\mathcal{D}_3) = \emptyset$.

(b) order-2 max VD. The curve is the bisector of disks $\mathcal{D}_1$ and $\mathcal{D}_3$. $\mathcal{V}(\mathcal{D}_1, \mathcal{D}_2) \neq \emptyset$, $\mathcal{V}(\mathcal{D}_2, \mathcal{D}_3) \neq \emptyset$.

Figure 2: Incremental construction does not apply as disks are contained inside other ones.

assumption does not always hold. As illustrated in Figure 2, although $\mathcal{D}_3$ is associated with no edge in the order-1 max VD of $S = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$, we have $\mathcal{V}^2(\{\mathcal{D}_2, \mathcal{D}_3\}, S) \neq \emptyset$.

In the paper, we propose an incremental algorithm to construct order-k max VD of disks. The intuition is as follows. Consider a Voronoi region, $\mathcal{V}^k(H, S)$, in an order-k max VD whose edge set is E. The generation of a new edge in E or the disappearance of an existing edge in E is referred to as an *event* of E. We observe that changing a disk's radius continuously makes some Voronoi vertices move along an identifiable trajectory while the others do not change. More importantly, disks' expansion does not necessarily induce an event of E. A disk may expand *continuously* but events happen *discretely*. This is illustrated in Figure 3. There are two kinds of vertices, i.e., *new* and *old*, denoted by circles and solid squares, respectively. As $\mathcal{D}_3$ expands, vertices of both kinds move along particular edges (arrows in the figures). We observe that solid squares move *away* their corresponding opposite vertex, while circles move *toward* their corresponding opposite vertex. Vertices may meet while moving. In this case, they may "destroy" an edge and create another one simultaneously. Additionally, a face may degenerate due to the meeting of moving vertices. Another face may be born simultaneously. In the following sections, we provide details of events that happen when a disk expands.

We discuss the changes of the diagram as disks expand in Section 3. Then, we propose an incremental algorithm to construct the order-k maximum Voronoi diagrams in Section 4. We conclude the paper in Section 5. We next introduce the concepts of order-k max VD in Section 2.

2 Preliminary

In the following discussion, we assume that no more than 2 disks' centers are co-linear, and no point in the plane is equal-distant to more than 3 disks under the $d_{\max}$ metric. We summarize the notations used throughout the paper as follows:

$\mathcal{D}_k(o_k, r_k)$: The disk centered at o_k with radius r_k. We use the notion $\mathcal{D}_k$ for simplicity.

S: The set of N disks $\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_N\}$.

S_i^ϵ: The modified S, in which the radius of $\mathcal{D}_i$ expands by a positive amount ϵ. $S_i^\epsilon = (S \setminus \{\mathcal{D}_i\}) \bigcup \{\mathcal{D}_{i'}(o_i, r_i + \epsilon)\}$.

H: A subset of S.

H_i^ϵ: The updated H. $H_i^\epsilon = (H \setminus \{\mathcal{D}_i\}) \bigcup \{\mathcal{D}_{i'}(o_i, r_i + \epsilon)\}$.

$d_{\max}(p, \mathcal{D}_i)$: The maximum distance from p to $\mathcal{D}_i$. $d_{\max}(p, \mathcal{D}_i) = d(p, o_i) + r_i$, where $d(\cdot, \cdot)$ is the Euclidean distance between two 2D points.

$\mathcal{V}_k(S)$: The max Voronoi face corresponding to disk $\mathcal{D}_k$ in the max VD of S. We simply refer to this as $\mathcal{V}_k$, when no confusion occurs.

$V(S)$: The max VD of the disk set S.

$\mathcal{V}^k(H, S)$: An order-k max Voronoi face associated with H, where $|H| = k$.

$V^k(S)$: The order-k max VD of S, or "diagram" for short. When $k = 1$, $V^k(S) \equiv V(S)$.

$v_{i,j,h}$: The max VD vertex corresponding to disks $\mathcal{D}_i$, $\mathcal{D}_j$, and $\mathcal{D}_h$.

$e_{i,j}$: The edge of the max VD corresponding to disks $\mathcal{D}_i$ and $\mathcal{D}_j$. $e_{i,j}$ is a hyperbola segment or an infinite hyperbola.

$b_{i,j}$: The locus of points p such that $d_{\max}(p, \mathcal{D}_i) = d_{\max}(p, \mathcal{D}_j)$. $b_{i,j}$ is a hyperbola with foci being o_i and o_j, or a straight line when $r_i = r_j$. We refer to $b_{i,j}$ as the bisector of o_i and o_j.

Two circles $\mathcal{D}_1(o_1, r_1)$ and $\mathcal{D}_2(o_2, r_2)$ are internally tangent if $d(o_1, o_2) = |r_1 - r_2|$. In the rest of the paper, by stating that a circle $\mathcal{D}_1$ is internally tangent to $\mathcal{D}_2$, we mean $\mathcal{D}_2$ lies interior to $\mathcal{D}_1$ unless stated otherwise. In addition, we say a circle $\mathcal{D}_1$ contains $\mathcal{D}_2$ if $\mathcal{D}_2$ lies interior to $\mathcal{D}_1$ but $\mathcal{D}_1$ is NOT internally tangent to $\mathcal{D}_2$.

We review two kinds of vertices described in [8]. Assume p is equal-distant to 3 disks under the $d_{\max}$ metric, i.e., $\mathcal{D}_i, \mathcal{D}_j$, and $\mathcal{D}_q$. Let C be the circle centered at p and internally tangent to the three disks. Assume that C contains $k-1$ other disks. By Definition 1, p belongs to 3 Voronoi faces of order k, namely, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i\}, S)$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j\}, S)$, and $\mathcal{V}^k(H \bigcup \{\mathcal{D}_q\}, S)$. In this case, p is referred to as a vertex of $V^k(S)$. Furthermore, p is also at the intersection of 3 Voronoi faces of order $k+1$, namely, $\mathcal{V}^{k+1}(H \bigcup \{\mathcal{D}_i, \mathcal{D}_j\}, S)$, $\mathcal{V}^{k+1}(H \bigcup \{\mathcal{D}_j, \mathcal{D}_q\}, S)$, and $\mathcal{V}^{k+1}(H \bigcup \{\mathcal{D}_q, \mathcal{D}_i\}, S)$. We say that, p is a *new vertex* of $V^k(S)$ and is an *old vertex* of $V^{k+1}(S)$. A new vertex of order-k Voronoi diagram becomes an old vertex in the order-$(k+1)$ Voronoi diagram; an old vertex of order-k Voronoi diagram does not exist in the next order Voronoi diagram.

To facilitate our discussion on disk expansion later, we introduce the notion of *pseudo disk*. A pseudo disk, denoted by $\mathcal{D}_\infty$, is a disk centered at the infinity with unit radius. Consider the infinite endpoint p of an infinite edge $e_{i,j}$ in an order-1 max VD. We have $d_{\max}(p, \mathcal{D}_j) = d_{\max}(p, \mathcal{D}_i) = \infty$. Therefore, we can associate p with a pseudo disk, that is, p is a vertex corresponding to 3 disks, namely, $\mathcal{D}_j$, $\mathcal{D}_i$, and $\mathcal{D}_\infty$. This way, we enclose the open end of each order-1 Voronoi face $\mathcal{V}(\{\mathcal{D}_i\}, S)$ with 2 edges, both associated with a pseudo disk. Therefore, all faces in any order-1 max VD are considered "closed". By similar arguments, the open end of the infinite edge $e_{i,j}$ is bounded by a new vertex corresponding to disks $\mathcal{D}_i$, $\mathcal{D}_j$, and $\mathcal{D}_\infty$.

We study the evolution of the order-k max VD as a disk expands. More specifically, we investigate 2 cases, namely, *i*) the expanding disk shares edges with at least one disk, and *ii*) the expanding disk does not share edges with any disk. We refer the disks of the first case as *type-I*, and the latter as *type-II*. Expanding a type-II disk eventually makes it a type-I, while expanding a type-I disk possibly makes some type-I disks type-II. In the following sections, we establish fundamental properties as a disk in an order-k max VD expands. We study type-I disks in section 3.1 and discuss in section 3.2 the expansion of a type-II disk.

Before proceeding, we introduce the notation of *maximum circumference*. Consider an edge $e_{i,j}$ of 2 faces $\mathcal{V}^k(H_1, S)$, $\mathcal{V}^k(H_2, S)$. There exists a circle that is internally tangent to $\mathcal{D}_i$ and $\mathcal{D}_j$, and contains $H_1 \bigcup H_2$. We refer to the circle as the *maximum circumference of* H_1 *and* H_2 associated with $e_{i,j}$, or simply maximum circumference. Similarly, a maximum circumference associated with a vertex is defined as the circle centered at the vertex and internally tangent to the disks constructing the vertex. If $e_{i,j}$ is an infinite edge, there exists a maximum circumference centered at infinity.

3 Geometrical analysis of order-k max VD

In this section, we study the changes of the order-k max VD as disks of type-I and type-II expand. The proofs are omitted due to space limit.

3.1 Expansion of a type-I disk

Expanding a disk $\mathcal{D}_i$, where $\mathcal{D}_i \in H$, makes some faces $\mathcal{V}^k(H, S)$ shrink.

Lemma 1 *Let* $\mathcal{V}^k(H, S)$ *and* $\mathcal{V}^k(H_i^\epsilon, S_i^\epsilon)$ *be the max Voronoi face of* H *and* H_i^ϵ *in* S *and* S_i^ϵ*, respectively. Then,* $\mathcal{V}^k(H_i^\epsilon, S_i^\epsilon) \subset \mathcal{V}^k(H, S)$.

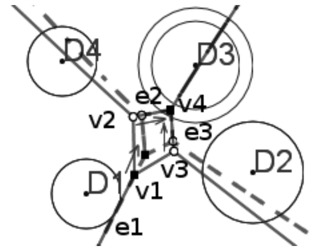


Figure 3: The change of order-2 max VD of four disks as disk $\mathcal{D}_3$ expands. Circles are old vertices, solid dots are new vertices. Dash curves are the diagram corresponding to the expanded $\mathcal{D}_3$. Vertices move as shown in arrows.

Roughly speaking, expanding a disk leads to changes in vertices, edges and faces of $V^k(S)$. Vertices move along edges and meet other ones. Formally,

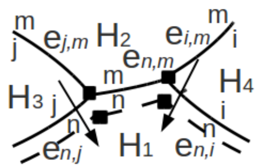
Lemma 2 *Consider a vertex* $v_{i,j,q}$ *of* $\mathcal{V}^k(H, S)$*. As* $\mathcal{D}_i$ *expands,* $v_{i,j,q}$ *moves along* $b_{j,q}$*. If* $v_{i,j,q}$ *is new, it moves away from the other end of* $e_{j,q}$ *elongating* $e_{j,q}$*. Otherwise, it moves towards the other end of* $e_{j,q}$ *to shorten* $e_{j,q}$.

This is illustrated in Figure 3. As $\mathcal{D}_3$ expands, the next vertex v_1 moves to elongate edge e_1, and old vertices v_2 and v_3 move to shorten edges e_2 and e_3, respectively. As a high order max VD evolves, different sequences of events occur as different types of vertices and edges are involved. The meeting of 2 new vertices or 2 old vertices results in an edge-death/birth, while the meeting of an old vertex and a new one additionally leads to face-death/birth.

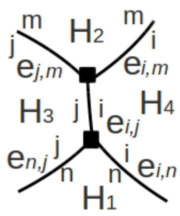
Lemma 3 *Let* $e_{i,j}$ *be an edge of* $\mathcal{V}^k(H, S)$ *with 2 new vertices, e.g.,* $v_{i,j-1,j}$ *and* $v_{i,j,j+1}$ *in counter-clockwise order in* $\mathcal{V}^k(H, S)$*.* $e_{i,j-1}$ *and* $e_{i,j+1}$ *are the edges of* $\mathcal{V}^k(H, S)$ *incident to* $v_{i,j-1,j}$ *and* $v_{i,j,j+1}$*, respectively. Let* p *be the intersection of* $b_{j,j-1}$ *and* $b_{j,j+1}$*. Assume that the next event as* $\mathcal{D}_i$ *expands is the meeting of* $v_{i,j-1,j}$ *and* $v_{i,j,j+1}$ *at* p*. If* $\mathcal{D}_{j-1} \neq \mathcal{D}_{j+1}$*, then with further expansion of* $\mathcal{D}_i$*,* $e_{i,j} = \emptyset$ *and* $e_{j-1,j+1} \neq \emptyset$*. In addition, both vertices of* $e_{j-1,j+1}$ *are new.*

Lemma 4 *Consider an edge* $e_{i,j}$ *of 2 faces* $\mathcal{V}^k(H_1, S)$ *and* $\mathcal{V}^k(H_2, S)$ *with 2 old vertices* $v_{i,j,n}$ *and* $v_{i,j,m}$*, where* $\{\mathcal{D}_i, \mathcal{D}_n, \mathcal{D}_m\} \subset H_1$ *and* $\{\mathcal{D}_j, \mathcal{D}_n, \mathcal{D}_m\} \subset H_2$*, respectively. Let* $\mathcal{V}^k(H_3, S)$ *and* $\mathcal{V}^k(H_4, S)$ *be the other faces, incident to* $v_{i,j,n}$ *and* $v_{i,j,m}$*, respectively. If* $\mathcal{D}_m \neq \mathcal{D}_n$*, and* $\mathcal{D}_n$ *expands such that the next event is the meeting of* $v_{i,j,n}$ *and* $v_{i,j,m}$*, further expansion of* $\mathcal{D}_n$ *results in i)* $e_{i,j} = \emptyset$*, and ii) "new" edge* $e_{n,m} \neq \emptyset$ *of* $\mathcal{V}^k(H_3, S)$ *and* $\mathcal{V}^k(H_4, S)$.

Lemmas 3 and 4 establish the changes in the diagram as vertices of the same kind meet. In general, the meeting of the 2 ends of an edge makes the edge disappear.

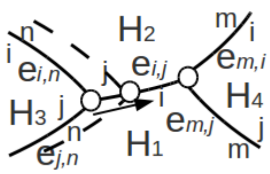


(a)

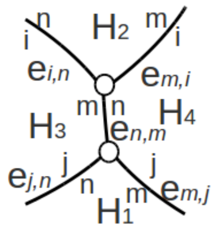


(b)

Figure 4: The evolution of edges $e_{i,j}$ and $e_{n,m}$ as new vertices move due to the expansion of $\mathcal{D}_n$.



(a)



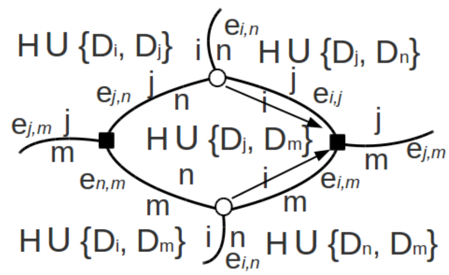
(b)

Figure 5: The evolution of edges $e_{i,j}$ and $e_{n,m}$ as old vertices move due to the expansion of $\mathcal{D}_n$.

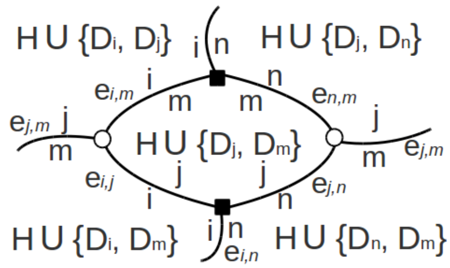
We refer to this as an *edge-death*. If the two vertices differ by one associated disk, another edge is born simultaneously. We refer to this as an *edge-birth*. Vertices of the newly born edge are new or old depending on the types of the meeting vertices. As two vertices are constructed by the same set of disks, the edge-death makes some face disappear. We refer to this as a *face-death* event. We skip the discussion on this kind of vertex meeting due to the space limit.

Figures 4 and 5 illustrate the results of Lemmas 3 and 4, respectively. Figures 4 shows the evolution of edge $e_{n,m}$ connecting 2 new vertices. Initially, $e_{n,m} \neq \emptyset$, and $e_{i,j} = \emptyset$ (Figure 4a). As $\mathcal{D}_n$ expands, 2 vertices of $e_{n,m}$ moves along the corresponding edges toward $\mathcal{V}^k(H_1, S)$ (arrow). If they meet, $e_{n,m}$ disappears and $e_{i,j}$ is born (Figure 4b). Both vertices are new. Figure 5 shows the evolution of edge $e_{i,j}$ connecting 2 old vertices. Initially, $e_{i,j} \neq \emptyset$ and $e_{n,m} = \emptyset$ (Figure 5a). As $\mathcal{D}_n$ expands, a vertex of $e_{i,j}$ moves toward the opposite end (arrow), which makes $e_{i,j}$ shrink. Eventually, $e_{i,j}$ degenerates as 2 vertices of $e_{i,j}$ meet, and $e_{n,m}$ is born (Figure 5b). Both vertices are old. Next, we present the change of the diagram as vertices of different kinds meet.

Lemma 5 *Assume that the new vertex $v_{i,j,n}$ meets the old one $v_{i,j,m}$ as $\mathcal{D}_n$ expands to $\mathcal{D}_{n'}(o_n, r_{n'})$, which results in the degeneration of edge $e_{i,j}$. The following holds: i) Either face incident to $e_{i,j}$, e.g., $\mathcal{V}^k(H_1, S)$, where $H_1 = H \bigcup \{\mathcal{D}_n, \mathcal{D}_i\}$ disappears; ii) Prior to the degeneration, $\mathcal{V}^k(H_1, S)$ shares edges $e_{j,n}$, $e_{m,n}$, $e_{m,i}$, and $e_{j,i}$ with only four neighbor faces $\mathcal{V}^k(H_2, S)$, $\mathcal{V}^k(H_3, S)$, $\mathcal{V}^k(H_4, S)$, and $\mathcal{V}^k(H_5, S)$, respectively, where $H_2 = H \bigcup \{\mathcal{D}_i, \mathcal{D}_j\}$, $H_3 = H \bigcup \{\mathcal{D}_i, \mathcal{D}_m\}$, $H_4 = H \bigcup \{\mathcal{D}_n, \mathcal{D}_m\}$, $H_5 = H \bigcup \{\mathcal{D}_j, \mathcal{D}_n\}$; and iii) Posterior to the degeneration of $\mathcal{V}^k(H_1, S)$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_m\}, S) \neq \emptyset$, and $e_{j,n}$, $e_{m,n}$, $e_{m,i}$, and $e_{j,i}$ of faces $\mathcal{V}^k(H_2, S)$, $\mathcal{V}^k(H_3, S)$, $\mathcal{V}^k(H_4, S)$, $\mathcal{V}^k(H_5, S)$ are replaced by $e_{i,m}$, $e_{i,j}$, $e_{n,j}$, $e_{n,m}$, respectively. Thus, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_m\}, S)$ consists of 4 edges, namely, $e_{i,m}$, $e_{i,j}$, $e_{n,j}$, and $e_{n,m}$, and 4 vertices, including 2 new vertices, namely, $v_{i,n,m}$ and $v_{i,j,n}$, and 2 old vertices, namely, $v_{i,j,m}$, and $v_{j,n,m}$.*



(a) The structure of the diagram prior to the disappearance of face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\})$.



(b) The structure of the diagram posterior to the disappearance of face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\})$.

Figure 6: The evolution of faces $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_j\})$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_m\}, S)$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_n, \mathcal{D}_m\}, S)$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_n\}, S)$, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\}, S)$, and $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_m\}, S)$ ($|H| = k - 1$). Squares denotes new vertices. Circles denotes old vertices.

Figure 6 illustrates the changes in the diagram as old vertices meet new vertices. Initially, $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\}, S) \neq \emptyset$ and $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_m\}, S) = \emptyset$. As $\mathcal{D}_n$ expands, old vertices $v_{i,j,n}$ and $v_{i,n,m}$ move along edges $e_{i,j}$ and $e_{i,m}$ (arrows), respectively, and meet new vertex $v_{i,j,m}$ making face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\}, S)$ disappear. Prior to its death, face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\}, S)$ shares 4 edges, i.e., $e_{i,j}, e_{n,j}, e_{n,m}$, and $e_{i,m}$ with faces $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_n\}, S), \mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_j\}, S), \mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_m\}, S)$, and $\mathcal{V}^k(H \bigcup \{\mathcal{D}_n, \mathcal{D}_m\}, S)$, respectively (Figure 6a). Posterior to the degeneration of face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_n\}, S)$, face $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_m\}, S)$ is born, which respectively shares 4 edges, i.e., $e_{m,n}, e_{m,i}, e_{j,i}$, and $e_{j,n}$ with faces $\mathcal{V}^k(H \bigcup \{\mathcal{D}_j, \mathcal{D}_n\}, S), \mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_j\}, S), \mathcal{V}^k(H \bigcup \{\mathcal{D}_i, \mathcal{D}_m\}, S)$, and $\mathcal{V}^k(H \bigcup \{\mathcal{D}_n, \mathcal{D}_m\}, S)$

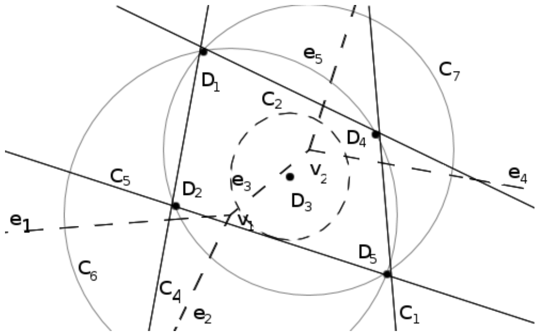


Figure 7: Illustration of a type-II disk's expansion. Dashed lines are the order-4 diagram of 5 points. Solid lines are maximum circumferences centered at infinite vertices. Light solid circles are the maximum circumferences centered at 2 finite vertices (v_1 and v_2).

(Figure 6b).

3.2 Expansion of a type-II disk

A disk $\mathcal{D}_i$ is a type-II disk if it does not shares any edge with other disks, i.e., $e_{i,j} = \emptyset$, $\forall \mathcal{D}_j \in S, j \neq i$. We can show that a disk $\mathcal{D}_i$ is type-II in $\mathcal{V}^k(S)$ when $\mathcal{D}_i$ contains k other disks, or $\mathcal{D}_i$ corresponds to all faces in $\mathcal{V}^k(S)$, i.e., $\mathcal{D}_i \in H$ for all $\mathcal{V}^k(H,S)$'s in $\mathcal{V}^k(S)$. In this section, we only discuss the latter case. It will be shown later that limiting our consideration to this case is sufficient to construct the order-k Voronoi diagrams. The basic idea is, we expand type-II disks to make them type-I, and then apply the techniques developed for type-I disks as discussed earlier. We can show that when a type-II disk expands, it only touches a maximum circumference centered at an infinite vertex. We illustrate the claim in Figure 7, which shows the order-4 Voronoi diagram of 5 disks of zero radius, $S = \{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_5\}$. The Voronoi regions are $\mathcal{V}^4(\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_3\}, S)$, $\mathcal{V}^4(\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_5, \mathcal{D}_3\}, S)$, $\mathcal{V}^4(\{\mathcal{D}_1, \mathcal{D}_5, \mathcal{D}_4, \mathcal{D}_3\}, S)$, and $\mathcal{V}^4(\{\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_3\}, S)$, which make $\mathcal{D}_3$ a type-II disk. The diagram (dashed) consists of 2 finite vertices, v_1 and v_2, whose corresponding maximum circumferences are C_6, and C_7 shown by light solid circles. The maximum circumferences centered at the infinite end of edges e_i are shown by straight lines C_i, $i \in \{1, 2, 3, 4\}$. As shown in the figure, as $\mathcal{D}_3$ expands (dashed circle), it first touches C_5, the maximum circumference centered at the infinite end of edge e_5 at a point in segment $\mathcal{D}_2\mathcal{D}_5$. In fact, we can show that:

Lemma 6 *It takes $O(k^2N)$ to process an order-k max VD of N disks so that it contains only type-I disks.*

We are now in the position to sketch an algorithm for constructing order-k maximum Voronoi diagrams of disks.

4 The incremental algorithm for order-k max VD construction

In constructing the order-k max VD of disks S, we start with an order-k Voronoi diagram (VD) of disk centers and iteratively expand each disk in S by a fixed amount $d_{\min}$, where $d_{\min} \triangleq \min_{i,j \in S} d(o_i, o_j) - \epsilon$. We stop when all disks reach their targeted size. The resulted diagram is the order-k max VD of S. Let $r_{\max} = \max_{i \in S} r_i$ and $r_{\min} = \min_{i \in S} r_i$. Clearly, the total number of rounds a disk needs to expand is bounded by $\lceil \frac{r_{\max} - r_{\min}}{d_{\min}} \rceil$. This implies that the algorithm terminates. Since disks expand by $d_{\min}$, they are not contained in other disks until they reach their targeted sizes. Furthermore, when a disk contains k other disks in its expansion, it becomes type-II since the k disks have reached their targeted sizes. Thus, its further expansion does not change the diagram. Therefore, the algorithm proceeds in such a way that all expanding disks are always type-I. As discussed in the previous sections, expanding disks do not make any new type-II disk. Thus, it is always possible to evaluate the expansion such that the next vertex meeting happens. The procedure of order-k max VD construction is summarized in Algorithm 1.

We first derive the number of edges and vertices in the order-k max VD by extending the results in [8].

Lemma 7 *The number of vertices and edges in an order-k max VD of N disks is $O(kN)$.*

Algorithm 1 takes the set S of N disks as inputs. It starts by constructing the order-k VD of disk centers (line 2), which in fact is the order-k max VD of disks whose radii are all equal to the minimum radius of N disks, denoted as S'. The process takes $O(k^2 N \log N)$ in running time ([8]). Since the order-k VD may contain type-II disks, we first make them type-I. This takes $O(k^2N)$. Then, we iteratively scan all disks in S and expand those whose radii are smaller than their respective sizes an amount $d_{\min}$ (lines 6 - 19). $d_{\min}$ can be computed in $O(N)$ using the order-1 VD of S', which is a byproduct in the incremental construction of $V^k(S')$. As each disk increases by $d_{\min}$, a disk $\mathcal{D}_i$ cannot contain disk $\mathcal{D}_j$ unless $\mathcal{D}_j$ has reached it targeted size. Therefore, it validates the earlier claim that we only need to consider type-II disks that do not contain other disks.

Theorem 1 *The order-k max VD of N disks can be constructed in $O\left(\left\lceil \frac{r_{\max} - r_{\min}}{d_{\min}} \right\rceil k^2 N \log N\right)$, where $r_{\max}$ and $r_{\min}$ are respectively the maximum and minimum radii of N disks, and $d_{\min}$ is the minimum distance between 2 disk centers.*

Proof. (Sketch) We analyze the expansion phase (lines 6-end). Let $\mathcal{D}_n$ be the disk that expands first, and $e_{n,m}$ be the edge that first disappears due to the expansion of $\mathcal{D}_n$ resulting in the birth of edge $e_{i,j}$ (Figure 4). When edge $e_{j,i}$ is born, the number of edges affected by the expansion of disks $\mathcal{D}_i$ and $\mathcal{D}_j$ increases by 2, while that of $\mathcal{D}_m$ decreases by 1. In general, when an edge of a face disappears, the total number of edges needed to be processed as other disks expand increases by 1. This observation also applies to two old vertices, as well as the case when an old vertex and a new vertex meet. In the order-k max VD, a face corresponds to k disks, thus each edge in a face may be processed k times. Since the number of edges in an order-k max VD is $O(kN)$, the total number of edges processed as N disks, each expands once, is $O(k^2N)$. Line 13 in Algorithm 1 requires a sorted list. Since the total number of edges processed is $O(k^2N)$, line 13 takes $O(k^2N \log N)$. Since a disk expands $d_{\min}$ in each round, it needs at most $\lceil \frac{r_{\max}-r_{\min}}{d_{\min}} \rceil$ expansions. Therefore, the time complexity of Algorithm 1 is $O\left(\left\lceil \frac{r_{\max}-r_{\min}}{d_{\min}} \right\rceil k^2 N \log N\right)$. □

Algorithm 1: Order-k Maximum Voronoi diagram of disks

input : A set of N disks $S = \{\mathcal{D}_1(o_1, r_1), \mathcal{D}_1(o_2, r_2), \ldots, \mathcal{D}_N(o_N, r_N)\}$

output: The order-k max VD of S, $V^k(S)$

```
1  r_min ← min_i r_i;
2  S' ← {D_1(o_1, r'_1 = r_min), ..., D_N(o_N, r'_N = r_min)};
3  Construct V^k(S');
4  process V^k(S') to transform type-II disks to type-I;
5  d_min ← min_{i,j∈S} d(o_i, o_j) − ε ;
6  repeat
7      foreach D_i such that r'_i < r_i do
8          if (r'_i + d_min) > r_i then
9              max_inc ← r'_i − r_i;
10         else
11             max_inc ← d_min;
12         while r'_i < max_inc do
13             find the smallest expansion e such that an
               event happens ;
14             if r'_i + e < max_inc then
15                 r'_i ← r'_i + e;
16                 update V^k(S') due to the event's
                   consequences;
17             else
18                 r'_i ← max_inc;
19                 re-calculate edges/vertices
                   corresponding to D_i;
20 until until all disks reach their targeted size ;
```

5 Conclusion

We have proposed an incremental algorithm to construct order-k maximum Voronoi diagram of disks in the plane. In our approach, disks iteratively expand from zero radius until they reach the targeted size, and the diagram is updated at certain sizes of disks. The algorithm runs in $O\left(\left\lceil \frac{r_{\max}-r_{\min}}{d_{\min}} \right\rceil k^2 N \log N\right)$ time, where $r_{\max}$ and $r_{\min}$ are respectively the maximum and minimum radii of N disks, and $d_{\min}$ is the minimum distance between 2 disk centers. Our contribution is two-fold. First, our algorithm provides a mechanism to quickly update the diagram of an order-k max VD as disk radii change (but disk centers are fixed). Second, our approach is amiable to distributed implementation. When a disk expands, it needs only information of the neighbors to update the diagram structure.

6 Acknowledgment

We thank the referees for providing constructive comments.

References

[1] F. Aurenhammer. Power diagrams: properties, algorithms and applications. *SIAM J. Comput.*, 16(1):78–96, 1987.

[2] F. Aurenhammer. Improved algorithms for disc and balls using power diagrams. *J. Algorithms*, 9(2):151–161, 1988.

[3] F. Aurenhammer, T. U. Graz, R. Klein, F. Hagen, and P. I. Vi. Voronoi diagrams. In *Handbook of Computational Geometry*, pages 201–290. Elsevier Science Publishers B.V. North-Holland.

[4] M. I. Karavelas and M. Yvinec. Dynamic additively weighted voronoi diagrams in 2d. pages 586–598, 2002.

[5] D.-S. Kim, D. Kim, and K. Sugihara. Voronoi diagram of a circle set from voronoi diagram of a point set: I. topology. *Computer Aided Geometric Design*, 18(6):541–562, 2001.

[6] D.-S. Kim, D. Kim, and K. Sugihara. Voronoi diagram of a circle set from voronoi diagram of a point set: Ii. geometry. *Computer Aided Geometric Design*, 18(6):563–585, 2001.

[7] D.-S. Kim, B. Lee, C.-H. Cho, and K. Sugihara. Plane-sweep algorithm of o(nlogn) for the inclusion hierarchy among circles. In *ICCSA (3)*, pages 53–61, 2004.

[8] D.-T. Lee. On k-nearest neighbor voronoi diagrams in the plane. *IEEE Trans. Comput.*, 31(6):478–487, 1982.

[9] M. I. Shamos. *Computational geometry.* PhD thesis, New Haven, CT, USA, 1978.

[10] K. Vu and R. Zheng. Robust coverage under uncertainty in wireless sensor networks. In *2011 Proceedings IEEE INFOCOM (INFOCOM 2011)*, pages 2015–2023, Shanghai, P.R. China, 4 2011.