Isoperimetric Triangular Enclosure with a Fixed Angle

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Abstract

Given a set S of $n > 2$ points in the plane (in general position), we show how to compute in $O(n^2)$ time, a triangle T with maximum (or minimum) area enclosing S among all enclosing triangles with fixed perimeter P and one fixed angle ω. We show that a similar approach can be used to compute a triangle T with maximum (or minimum) perimeter enclosing S among all enclosing triangles with fixed area A and one fixed angle ω.

1 Introduction

The classical isoperimetric problems are

1. Of all plane figures of equal area, what is the one with minimum perimeter?

2. Of all plane figures of equal perimeter, what is the one with maximum area?

Several related problems and a complete discussion on the foundations and applications of these problems together with the proof of the following result can be found in Polya [7].

Theorem 1 (Isoperimetric Theorem)

1. *Of all plane figures of equal area, the circle has minimum perimeter.*

2. *Of all plane figures of equal perimeter, the circle has maximum area.*

3. *1 and 2 are equivalent.*

4. *Let $n > 2$ be a fixed integer. Of all n-gons of equal area, the regular n-gon has minimum perimeter.*

5. *Let $n > 2$ be a fixed integer. Of all n-gons of equal perimeter, the regular n-gon has maximum area.*

Given a fixed area (respectively perimeter), there is no upper bound (respectively lower bound) on the perimeter (respectively area) a plane figure can have. In this paper, we are interested in figures that enclose a set of at least 3 non-collinear points. Then it is relevant to maximize the perimeter given a fixed area (respectively minimize the area given a fixed perimeter). We refer to these four isoperimetric problems as **FIP**.

Let ω be a fixed angle with $0 < \omega < \pi$. A triangle that has an angle ω is called an ω-triangle. In this paper, we study the **FIP** with two additional constraints: (1) the plane figures we consider are ω-triangles, and (2) they must enclose a given set S of n points.

These problems are a variant of the problems studied in [1, 2, 3, 4, 5, 6, 8, 9]. Most of these problems can be solved in linear time when the input is a convex n-gon or in $O(n \log n)$ time when the input is a set of n points because of an interspersing lemma proper to each of these problems. Essentially, an interspersing lemma states that given a local extremum, if we turn clockwise around the convex hull of the set of points, then we will find all the other local extrema also in clockwise order (there is no need to backtrack). So it takes $O(n \log n)$ time to compute the convex hull, then it takes $O(n)$ time to compute one local extremum, then it takes $O(n)$ time to compute all the other local extrema and finally, it takes $O(n)$ time to compute the global extrema. Unfortunately, such a lemma does not hold in the isoperimetric case. Our solution to the **FIP** takes $O(n^2)$ time. We explain in Section 5 why the canonical interspersing lemma does not apply, though we do not have a proof of a quadratic lower bound. The **FIP** can also be compared to the following problem (see [7], p.180): "Given an angle (the infinite part of a plane between two rays drawn from the same initial point). Find the maximum area cut off from it by a line of given length.". Note that if the fixed perimeter or the fixed area is too small, no solution exists.

2 Preliminary Results

Let $T = \triangle bqc$ be an ω-triangle with $\angle bqc = \omega$. Denote the area and the perimeter of T by A and P respectively. Let $x = |bq|$, $y = |qc|$ and $z = |bc|$. Therefore,

$$\begin{aligned} P &= x + y + z \ , \\ A &= \frac{1}{2} xy \sin(\omega) \ , \\ z^2 &= x^2 + y^2 - 2xy \cos(\omega) \ , \end{aligned}$$

from which

$$x = \frac{P^2 \sin(\omega) + 4A(1 + \cos(\omega))}{4P \sin(\omega)} \tag{1}$$

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$$
\begin{aligned}
& -\frac{\sqrt{(P^2\sin(\omega)+4A(1+\cos(\omega)))^2-32AP^2\sin(\omega)}}{4P\sin(\omega)} \ , \\
y &= \frac{P^2\sin(\omega)+4A(1+\cos(\omega))}{4P\sin(\omega)} \\
& +\frac{\sqrt{(P^2\sin(\omega)+4A(1+\cos(\omega)))^2-32AP^2\sin(\omega)}}{4P\sin(\omega)} \ , \\
z &= \frac{P^2\sin(\omega)-4A(1+\cos(\omega))}{2P\sin(\omega)} \ .
\end{aligned}
$$

Then, from (1), we have

$$
\begin{aligned}
A(x) &= \frac{Px\sin(\omega)(P-2x)}{4(P-x(1+\cos(\omega)))} \ , \\
P(x) &= \frac{2A+x^2\sin(\omega)}{x\sin(\omega)} \\
&+ \frac{\sqrt{4A^2+x^4\sin^2(\omega)-4x^2A\sin(\omega)\cos(\omega)}}{x\sin(\omega)} \ .
\end{aligned}
$$

With standard calculus techniques, we can prove the following properties (refer to Subsection 2.1). If P is fixed, then $A(x)$ is defined for all $0 < x < \frac{1}{2}P$. It is increasing for $x \in \left]0, \frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P\right]$ and decreasing for $x \in \left[\frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P, \frac{1}{2}P\right[$. Thus the area is a unimodal function of x. When $x = \frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P$, T is isosceles with $x = y$. As for $P(x)$, if A is fixed, then it is defined for all $x > 0$. It is decreasing for $x \in \left]0, \frac{\sqrt{2}\sqrt{A}}{\sqrt{\sin(\omega)}}\right]$ and increasing for $x \in \left[\frac{\sqrt{2}\sqrt{A}}{\sqrt{\sin(\omega)}}, \infty\right[$. Thus the perimeter is a unimodal function of x. When $x = \frac{\sqrt{2}\sqrt{A}}{\sqrt{\sin(\omega)}}$, T is isosceles with $x = y$.

From the previous discussion, we see that the **FIP** can be solved by focusing on the length of one of the sides of the ω-triangle. The angle ω of the desired ω-triangle is part of an ω-wedge.

Definition 1 (ω-Wedge) *Let q be a point in the plane. Let Δ_1 and Δ_2 be two rays emanating from q such that the smallest angle between Δ_1 and Δ_2 is ω. The closed set formed by q, Δ_1, Δ_2 and the points between Δ_1 and Δ_2 is an* ω-wedge, *denoted $\mathcal{W}(\omega, q, \Delta_1, \Delta_2)$. The point q is the* apex *of the ω-wedge. An ω-wedge W is said to* **enclose** *a convex n-gon Q when $Q \subseteq W$ and both Δ_1 and Δ_2 are tangent to Q.*

Therefore the vertex q of the desired ω-wedge is on an ω-cloud.

Definition 2 (ω-Cloud) *By rotating an enclosing ω-wedge around a convex n-gon Q, the apex traces a sequence of circular arcs. This sequence is called an* ω-cloud, *denoted Ω (refer to Figure 1). The circular arcs of Ω are labelled in clockwise order by Γ_j for $0 \leq j \leq n'-1$. We note that $n' = O(n)$ [3].*

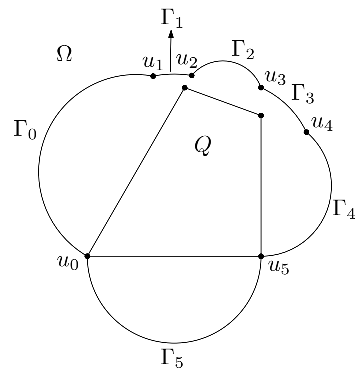


Figure 1: Ω is the $\frac{1}{2}\pi$-cloud of Q.

The proof of the following Lemma is similar to the proof of Lemma 1 in [1].

Lemma 2 *Let A and P be two positive real numbers. Take $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$.*

1. *Consider the set of ω-triangles $\triangle bqc$ with perimeter P such that $b \in \Delta_1$ and $c \in \Delta_2$. The side bc of these ω-triangles are tangent to a common circle with radius $r_P = \frac{1}{2}P\tan\left(\frac{1}{2}\omega\right)$ (refer to Figure 2). We call this circle the* perimeter circle of W *and we denote it by $\mathcal{C}_P$. The center of $\mathcal{C}_P$ is on the angle bisector of W and Δ_1 (respectively Δ_2) is tangent to $\mathcal{C}_P$ at t_1 (respectively at t_2) where $|qt_1| = \frac{1}{2}P$ (respectively $|qt_2| = \frac{1}{2}P$).*

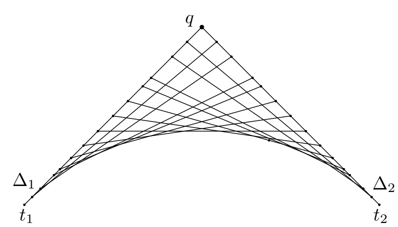


Figure 2: A $\frac{1}{2}\pi$-wedge together with $\frac{1}{2}\pi$-triangles $\triangle bqc$ with perimeter P such that $b \in \Delta_1$ and $c \in \Delta_2$.

2. *Consider the set of ω-triangles $\triangle bqc$ with area A such that $b \in \Delta_1$ and $c \in \Delta_2$. The sides bc of all these ω-triangles are tangent to a common hyperbola with asymptotes Δ_1 and Δ_2 (refer to Figure 3). We call this hyperbola the* area hyperbola of W *and we denote it by $\mathcal{H}_A$. The center of $\mathcal{H}_A$ is on q.*

In this paper, we explain in detail how to find an ω-triangle of minimum and maximum area with fixed perimeter. The solution when the area is fixed is almost identical since both $A(x)$ and $P(x)$ are unimodal functions. We use a technique similar to the one of Bose

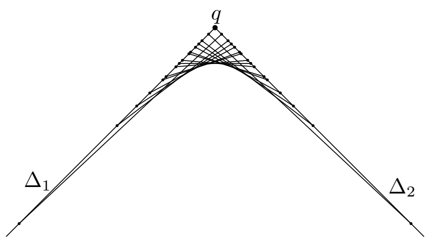


Figure 3: A $\frac{1}{2}\pi$-wedge together with $\frac{1}{2}\pi$-triangles $\triangle bqc$ with area A such that $b \in \Delta_1$ and $c \in \Delta_2$.

and De Carufel in [2]. The main difference is in the lack of interspersing lemma (refer to Section 5). If the input is a set S of $n \geq 3$ non-collinear points, we first compute the convex hull of S. In what follows, we show how to solve the **FIP** when the input is a convex n-gon Q. Moreover, $Q = \text{int}(Q) \cup \delta Q$, where $\text{int}(Q)$ is the interior of Q and δQ is the boundary of Q.

2.1 Analysis of $A(x)$ and $P(x)$

Given

$$A(x) = \frac{Px\sin(\omega)(P-2x)}{4(P-x(1+\cos(\omega)))}$$

for $0 < x < \frac{1}{2}P$, we have

$$A'(x) = \frac{P\sin(\omega)(P^2 - 4Px + 2x^2 + 2x^2\cos(\omega))}{4(P-x-x\cos(\omega))^2}\ .$$

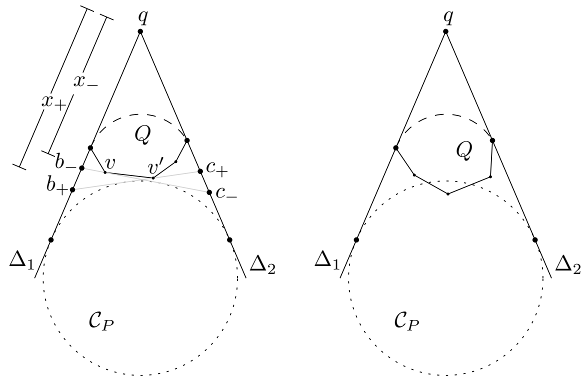
Hence, $A'(x) = 0$ if and only if $x = \frac{2\pm\sqrt{2-2\cos(\omega)}}{2(1+\cos(\omega))}P$. We reject $x = \frac{2+\sqrt{2-2\cos(\omega)}}{2(1+\cos(\omega))}P$ because

$$\begin{aligned}
\frac{2+\sqrt{2-2\cos(\omega)}}{2(1+\cos(\omega))}P &= \frac{1+\sqrt{\frac{1-\cos(\omega)}{2}}}{1+\cos(\omega)}P \\
&= \frac{1+\sin\left(\frac{1}{2}\omega\right)}{1+\cos(\omega)}P \\
&> \frac{1}{2}P \qquad (0<\omega<\pi).
\end{aligned}$$

As for $x = \frac{2-\sqrt{2-2\cos(\omega)}}{2(1+\cos(\omega))}P$, it is a maximum because

$$\begin{aligned}
A''(x) &= -\frac{P^3\sin(\omega)(1-\cos(\omega))}{(P-x-x\cos(\omega))^3} \\
&< -\frac{P^3\sin(\omega)(1-\cos(\omega))}{\left(P-\frac{1}{2}P-\frac{1}{2}P\cos(\omega)\right)^3} \\
&= -\frac{8\sin(\omega)}{(1-\cos(\omega))^2} \\
&< 0 \qquad (0<\omega<\pi).
\end{aligned}$$

Therefore, if P is fixed, $A(x)$ is increasing for $x \in \left]0, \frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P\right]$ and decreasing for $x \in \left[\frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P, \frac{1}{2}P\right[$. The analysis of $P(x)$ is similar.



(a) An enclosing ω-triangle exists because $\text{int}(Q) \cap \text{int}(\mathcal{C}_P) = \emptyset$. b_- is such that $x_- = |qb_-|$ is the smallest. b_+ is such that $x_+ = |qb_+|$ is the longest.

(b) No enclosing ω-triangle exists because $\text{int}(Q) \cap \text{int}(\mathcal{C}_P) \neq \emptyset$.

Figure 4: In Figure 4(a), an enclosing ω-triangle exists. In Figure 4(b), no enclosing ω-triangle exists.

3 The Solution for a Fixed ω-Wedge

Take a convex polygon Q and an ω-wedge $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$ enclosing Q (refer to Figure 4(a)). The solution for W is based on the following observations.

Observation 1

1. *An enclosing ω-triangle T with perimeter P can be constructed on W if and only if $\text{int}(Q) \cap \text{int}(\mathcal{C}_P) = \emptyset$ (refer to Figure 4).*
2. *There exists exactly one T if and only if Q and $\mathcal{C}_P$ are tangent.*
3. *Suppose that more than one T exists.*
   (a) *We have to compare the ω-triangle $\triangle b_-qc_-$ with the smallest side $x_- = |qb_-|$ and the ω-triangle $\triangle b_+qc_+$ with the longest side $x_+ = |qb_+|$ to find the one with minimum area. These two candidates are such that b_-c_- and b_+c_+ are tangent to both Q and $\mathcal{C}_P$ (refer to Figure 4(a)). Let v_- (respectively v_+) be the vertex of Q such that b_-c_- (respectively b_+c_+) is tangent to Q at v_- (respectively at v_+). We say that v_- and v_+ are* witness vertices. *If b_-c_- (respectively b_+c_+) is flush with an edge e_- (respectively e_+) of Q, we define v_- (respectively v_+) as the vertex on $e_- = Q \cap b_-c_-$ (respectively on $e_+ = Q \cap b_+c_+$) that is the closest to b_- (respectively to b_+).*

(b) Any enclosing ω-triangle $\triangle bqc$ with perimeter P strictly between $\triangle b_-qc_-$ and $\triangle b_+qc_+$ is such that bc is tangent to $\mathcal{C}_P$ and bc does not touch Q.

(c) If one of these T's is isosceles, then it has maximum area. One of the T's is isosceles if and only if $x_- \leq \frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))}P \leq x_+$. Otherwise, one of $\triangle b_-qc_-$ and $\triangle b_+qc_+$ has minimum area and the other one has maximum area by unimodality of $A(x)$.

Hence, if there is no witness vertex, then no ω-triangle T can be constucted on W. If $v_- = v_+$, then there exists exactly one such T. If $v_- \neq v_+$, then there exists infinitely many such T's.

For a given ω-wedge W and a given vertex v of Q, it takes $O(1)$ time to decide whether or not $v = v_-$ and whether or not $v = v_+$ (refer to Subsection 3.1). Thus, for a given ω-wedge W, it takes $O(n)$ time to compute v_-, v_+, x_-, x_+, $\triangle b_-qc_-$ and $\triangle b_+qc_+$.

3.1 Decide Whether a Vertex is a Witness Vertex

Let $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$ be a fixed ω-wedge enclosing a convex n-gon Q and v be a vertex of Q. Denote by Γ the circular arc of Ω such that $q \in \Gamma$. In this subsection, we explain how to decide whether v is a witness vertex of W.

Without loss of generality, Γ is the locus of points q such that $\angle v_iqv_j = \omega$, where v_i and v_j are two vertices of Q (refer to Figure 5). Hence we can take $v_i = (0, 0)$,

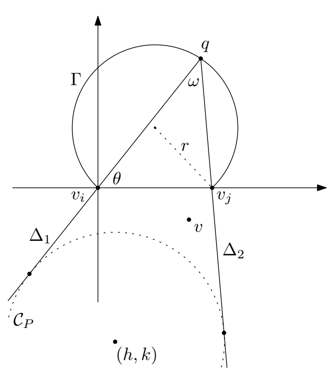


Figure 5: A formula for the center (h, k) of $\mathcal{C}_P$.

and $v_j = (2r\sin(\omega), 0)$, where r is the radius of Γ. Let $\theta = \angle v_jv_iq$. In this setting, the radius of $\mathcal{C}_P$ is $r_P = \frac{1}{2}P\tan\left(\frac{1}{2}\omega\right) = \frac{P\sin(\omega)}{2(1+\cos(\omega))}$ by Lemma 2-1 and the center (h, k) of $\mathcal{C}_P$ is such that

$$
\begin{aligned}
h &= \frac{1}{2(1+\cos(\omega))} \times \\
&\quad \big(6r\cos^2(\omega)\sin(\theta)\cos(\theta) + 4r\cos(\omega)\sin(\theta)\cos(\theta) \\
&\quad +6r\sin(\omega)\cos(\omega)\cos^2(\theta) - P\cos(\omega)\cos(\theta) \\
&\quad -2r\sin(\omega)\cos(\omega) + P\sin(\omega)\sin(\theta) \\
&\quad +2r\sin(\omega)\cos^2(\theta) - 2r\sin(\theta)\cos(\theta) \\
&\quad -P\cos(\theta) + 2r\sin(\omega)\big) \ , \\
k &= -\cot(\theta)h \\
&\quad -\frac{P - 4r\cos(\omega)\sin(\theta) - 4r\sin(\omega)\cos(\theta)}{2\sin(\theta)} \ .
\end{aligned}
$$

These formulas can be obtained by analytic geometry and Lemma 1 in the following way. By geometry and trigonometry, we have $q = (2r\cos(\theta)\sin(\theta+\omega), 2r\sin(\theta)\sin(\theta+\omega))$. Since (h, k) is on the angle bisector of W and $r_P = \frac{P\sin(\omega)}{2(1+\cos(\omega))}$, then the distance between (h, k) and Δ_1 and the distance between (h, k) and Δ_2 are equal to $\frac{P\sin(\omega)}{2(1+\cos(\omega))}$. Hence, we can find the equation of Δ_1, Δ_2 and the angle bisector of W. From these equations, we deduce the formulas for h and k.

Let e and e' be the two edges adjacent to v. Denote by Δ_e and $\Delta_{e'}$ the lines through e and e' respectively. If $e \cap \mathrm{int}(\mathcal{C}_P) \neq \emptyset$ or $e' \cap \mathrm{int}(\mathcal{C}_P) \neq \emptyset$, then v is not a witness vertex. Moreover, no enclosing ω-triangle can be constructed on W. It follows from Observation 1-1. Suppose that $e \cap \mathrm{int}(\mathcal{C}_P) = \emptyset$ and $e' \cap \mathrm{int}(\mathcal{C}_P) = \emptyset$.

- If $\mathcal{C}_P \cap \Delta_e = \emptyset$ and $\mathcal{C}_P \cap \Delta_{e'} = \emptyset$, then v is not a witness vertex.
- If $\mathcal{C}_P \cap \Delta_e \neq \emptyset$ or $\mathcal{C}_P \cap \Delta_{e'} \neq \emptyset$, then v is a witness vertex.

It all follows from Lemma 2-1 and Observation 1-3a. Since we supposed that W is fixed, then r, ω and θ are fixed. So all these tests can be done in $O(1)$ time.

4 Turning Around the ω-Cloud

Let v_- and v_+ be the two witness vertices (possibly $v_- = v_+$) of an ω-wedge $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$ enclosing Q. Let Γ_j be the circular arc of Ω such that $q \in \Gamma_j$. As q moves on Γ_j, $\triangle b_-qc_-$ and $\triangle b_+qc_+$ move continuously around Q. Moreover, there is a circular arc $\Gamma'_j \subseteq \Gamma_j$ (that can be reduced to a single point) for which the witness vertices remain v_- and v_+. We say that v_- and v_+ are *persistent witness vertices* of Γ'_j. Among all the enclosing ω-triangles that can be constructed on the enclosing ω-wedges having their apex on Γ'_j, we find the one with the smallest $x_- = |b_-c_-|$ (respectively with the longest $x_+ = |b_+c_+|$). We denote this triangle by $T_{\min} = \triangle qb_{\min}c_{\min}$ (respectively by $T_{\max} = \triangle qb_{\max}c_{\max}$) and we let $x_{\min} = |b_{\min}c_{\min}|$ (respectively $x_{\max} = |b_{\max}c_{\max}|$). By continuity, for all x such that $x_{\min} \leq x \leq x_{\max}$, there exists an enclosing ω-wedge with apex on Γ'_j such that an enclosing ω-triangle $\triangle bqc$ can be constructed with $x = |bc|$. Therefore, on Γ'_j,

- the minimum enclosing area ω-triangle that can be constructed is either $\triangle qb_{\min}c_{\min}$ or $\triangle qb_{\max}c_{\max}$.

- – If $x_{\min} \leq \frac{2-\sqrt{2}\sqrt{1-\cos(\omega)}}{2(1+\cos(\omega))} P \leq x_{\max}$, then an isosceles ω-triangle with perimeter P has maximum area.
  – Otherwise, one of $\triangle q b_{\min} c_{\min}$ and $\triangle q b_{\max} c_{\max}$ has minimum area and the other one has maximum area.

Given an enclosing ω-wedge $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$ with $q \in \Gamma_j$, it takes $O(n)$ time to compute v_-, v_+, x_-, x_+, $\triangle b_- q c_-$ and $\triangle b_+ q c_+$ (refer to Section 3). Then it takes $O(1)$ time to compute $\Gamma'_j \subseteq \Gamma_j$ such that Γ'_j has persistent witness vertices v_- and v_+ (refer to Subsection 4.1). Then it takes $O(1)$ time to compute $x_{\min}$, $x_{\max}$, $T_{\min}$ and $T_{\max}$ (refer to Subsection 4.2).

Once we solved the **FIP** for Γ'_j, we need to compute the next circular arc together with its persistent witness vertices. Denote the next circular arc by Γ''_j Note that since $\Gamma'_j \subseteq \Gamma_j$, then either $\Gamma''_j \subset \Gamma_j$ or $\Gamma''_j \subseteq \Gamma_{j+1}$. Two different events can happen:

Event 1 Γ''_j has no persistent witness vertex

Event 2 or Γ''_j has persistent witness vertices and at least one of v_- and v_+ is different from the persistent witness vertices of Γ'_j.

If Γ''_j has persistent witness vertices, then by continuity, these persistent witness vertices are adjacent to or equal to the persistent witness vertices of Γ'_j. So there are 8 pairs of vertices to test for persistence (recall that at least one of v_- and v_+ is not the same compared to Γ'_j). If none of these 8 pairs is a pair of persistent witness vertices for Γ''_j, then Γ''_j has no persistent witness vertex. Therefore, both **Event 1** and **Event 2** can be detected in $O(1)$ time (refer to Subsection 4.1).

With this technique, we subdivide the circular arcs Γ_j of Ω into subarcs that either have persistent witness vertices or not. Given a subarc Γ'_j, we explained how to solve the **FIP** on Γ'_j in $O(n)$. Given the witness vertices of Γ'_j, we also explained how to solve the **FIP** on every next subarc in $O(1)$. How many of these subarcs are there? In Section 5, we present an example that shows that Γ_j can be subdivided into a linear number of subarcs. However, we do not know whether a constant number or a linear number of circular arcs of Ω can be subdivided into a linear number of subarcs. So the lower bound on the time of computation of the solution to the **FIP** remains an open question.

4.1 Computing Γ'_j

Given two vertices v_- and v_+ and a circular arc Γ_j of Ω, we explain how to find $\Gamma'_j \subseteq \Gamma_j$ such that Γ'_j has v_- and v_+ as persistent witness vertices. From the discussion of Subsection 3.1, Lemma 2-1 and Observation 1, we need to find the values of θ such that Δ_e is tangent to $\mathcal{C}_P$ and the values of θ such that $\Delta_{e'}$ is tangent to $\mathcal{C}_P$. We explain how to find the values of θ such that Δ_e is tangent to $\mathcal{C}_P$ (the values of θ such that $\Delta_{e'}$ is tangent to $\mathcal{C}_P$ can be found in a similar way). Take $\mathcal{C}_P : (x-h)^2 + (y-k)^2 = r_P^2$ and $\Delta_e : y = \mu x + \lambda$. From analytic geometry, Δ_e is tangent to $\mathcal{C}_P$ if and only if

$$k \pm \frac{r_P}{\sqrt{\mu^2+1}} = \mu \left(h - \frac{\mu r_P}{\sqrt{\mu^2+1}} \right) + \lambda \ . \tag{2}$$

Since μ, λ, P, ω and r are constant, (2) is an equation of degree 2 in $\sin(\theta)$ and $\cos(\theta)$. Therefore, it can be transformed into an equation of degree 4 in $\sin(\theta)$. If (2) has no solution in θ or if the solutions are not sound with respect to Γ_j, then there is no $\Gamma'_j \subseteq \Gamma_j$ that has v_- and v_+ as persistent witness vertices. Thus it can be solved exactly in $O(1)$ time.

4.2 Compute $b_{\min}$ and $b_{\max}$

As we did in Section 3.1, let $W = \mathcal{W}(\omega, q, \Delta_1, \Delta_2)$ be a fixed ω-wedge enclosing a convex n-gon Q. Let Γ be a circular arc such that v is a persistent witness vertex of Γ. Without loss of generality, Γ is the locus of points q such that $\angle v_i q v_j = \omega$, where v_i and v_j are two vertices of Q (refer to Figure 5). Hence we can take $v_i = (0,0)$, and $v_j = (2r\sin(\omega), 0)$, where r is the radius of Γ. Let $\theta = \angle v_j v_i q$.

Let $b = (\alpha, \alpha\tan(\theta)) \in \Delta_1$ be a point such that $\triangle bqc$ has the prescribed perimeter, where c is the intersection point of the line through bv and the line through qv_j. Therefore, the line $\Delta : y = \mu x + \lambda$ through bv satisfies (2) from the discussion of Subsection 4.1. For a fixed θ, it is an equation in α. Hence, in order to find $b_{\min}$ or $b_{\max}$, we need to optimize $|qb|$ subject to (2). It leads to a polynomial equation in $\sin(\theta)$ of high degree. This can be done with numerical methods. For a given fixed error tolerance, it takes $O(1)$ time to compute $b_{\min}$ or $b_{\max}$.

5 An Interspersing Lemma

If we translate the interspering lemmas of [1, 2, 3, 6] in terms of the **FIP**, we get the following statement: "as q turns clockwise around Ω, v_- and v_+ turn clockwise around Q." This statement implies that the time of computation of the solution to the **FIP** is $O(n)$ (when the input is a convex n-gon) since we only need to go around once. Unfortunately, this statement is false in the current setting. In this section, we construct an example where q turns clockwise around Ω and v_+ turns counter-clockwise around Q. Because of this example, the time of computation of our algorithm is $O(n^2)$. Indeed, this example suggests that all circular arcs Γ_j of Ω could be subdivided into a linear number of subarcs (refer to Section 4).

Consider the example of Figure 6 where $\omega = \frac{1}{2}\pi$. Four vertices of Q appear, namely v_i, v_k, $v_{i'}$ and $v_{i'+1}$. The circular arc Γ_j of Ω is built over v_i and v_k, and we consider an enclosing $\frac{1}{2}\pi$-wedge $W = \mathcal{W}(\frac{1}{2}\pi, q, \Delta_1, \Delta_2)$ where $q \in \Gamma_j$. $T_+ = \triangle b_+ q c_+$ is such that b_+c_+ is flush

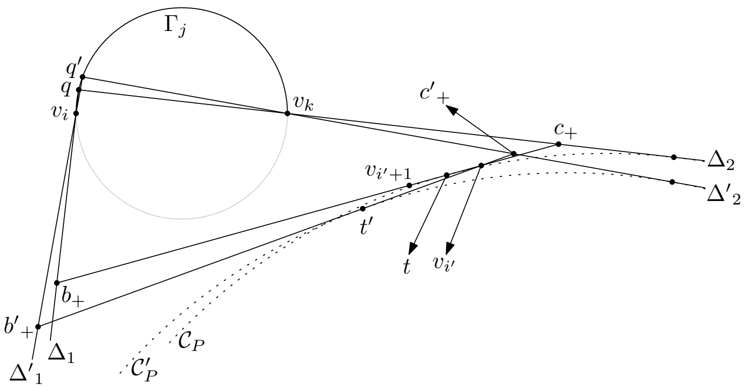


Figure 6: As q turns clockwise around Ω, v_+ turns counter-clockwise around Q.

with the edge $e_{i'} = v_{i'}v_{i'+1}$ of Q and b_+c_+ is tangent to $\mathcal{C}_P$ at $t \in e_{i'}$. Therefore, the witness vertex v_+ of T_+ is $v_+ = v_{i'+1}$.

Let $W' = \mathcal{W}(\frac{1}{2}\pi, q', \Delta'_1, \Delta'_2)$ be an enclosing $\frac{1}{2}\pi$-wedge obtained by a clockwise rotation of q around Γ_j and such that $v_{i'} \notin \Delta'_2$. $T'_+ = \triangle b'_+ q' c'_+$ is such that $b'_+c'_+$ touches Q at $v_{i'}$ and $b'_+c'_+$ is tangent to $\mathcal{C}'_P$ at $t' \notin Q$. Therefore, $v'_+ = v_{i'}$. Hence, this is an example where q turns clockwise around Ω and v_+ turns counter-clockwise around Q.

Using the same strategy, we can make v_+ turn counter-clockwise around Q and visit m vertices for any $m \geq 1$. Let $q_0 = q, q_1, ..., q_{m-1} = q' \in \Gamma_j$ be a sequence of m different points from q to q'. For each q_l $(0 \leq l \leq m-1)$, consider the wedge $W_l = \mathcal{W}(\frac{1}{2}\pi, q_l, \Delta_{l,1}, \Delta_{l,2})$ and $T_{l,+} = \triangle b_{l,+} q_l c_{l,+}$. Put a vertex $v_{i'-l}$ on $b_{l,+}c_{l,+}$ such that $b_{l,+}c_{l,+}$ is flush with the edge $e_{i'-l} = v_{i'-l}v_{i'-l+1}$ and $v_{i'-l}$ is strictly between $v_{i'-l+1}$ and $c_{l,+}$. This way, $v_{l,+} = v_{i'-l+1}$ so as q turns clockwise around Ω, v_+ turns counter-clockwise around Q and visits m vertices for any $m \geq 1$.

This proves that the canonical interspersing lemma for the **FIP** does not stand. However, it does not prove that the lower bound on the the time of computation of the solution to the **FIP** is $\Omega(n^2)$. The construction we presented works for Γ_j, but we do not know if it is possible to do such a construction on all the circular arcs of Ω simultaneously. This question remains open.

6 Conclusion

We explained in detail how to find an ω-triangle of minimum and maximum area with fixed perimeter. Our solution takes $O(n^2)$. If one fixes the area rather than the perimeter, a similar solution exists by switching the word "perimeter" with "area", "minimum" with "maximum", and "perimeter circle" with "area hyperbole".

Two main questions remain open about the **FIP**. Is $\Omega(n^2)$ the lower bound on the time of computation of the solution to the **FIP**? Is it possible to simplify the polynomial equations involved in the computation of $b_{\min}$ and $b_{\max}$? As for more general open questions related to the **FIP**,

1. What is the time of computation of the solution to the **FIP** when there is no angle constraint?

2. What is the time of computation of the solution to the **FIP** when we consider shapes with curved boundary?

3. What is the solution in three dimensions?

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