On Fence Patrolling by Mobile Agents

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Abstract

Suppose that a fence needs to be protected by k mobile agents with maximum speeds $v_1, \ldots, v_k$ so that each point on the fence is visited by some agent within every duration of a predefined time. The problem is to determine if this requirement can be met, and if so, to design a suitable schedule for the agents. Alternatively, one would like to find a schedule that minimizes the *idle time*, that is, the longest time interval during which some point is not visited by any agent. The problem was introduced by Czyzowicz et al. (2011). We revisit this problem and discuss several strategies for the cases of open and respectively closed fence.

1 Introduction

A set of mobile agents with predefined (possibly distinct) maximum speeds are in charge of *guarding* or in other words *patrolling* a given region of interest. Two interesting uni-dimensional variants where the agents move along a curve (e.g., the boundary of the region), have been introduced by Czyzowicz et al. [1]: (i) only part of the boundary, that is, an open curve, or *open fence*, needs to be guarded; (ii) the entire boundary, that is, a closed curve (cycle), or *closed fence*, needs to be guarded. For simplicity (and without loss of generality) it can be assumed that the open curve is a segment and the closed curve is a circle.

Given a schedule of the agents over some time interval $[0, t]$, the *idle time* I is the longest time interval during which a point of the fence remains unvisited, taken over all points. We are interested in guarding over an unlimited time interval, i.e., over the interval $[0, \infty)$. If the schedule of the agents is such that the positions of the agents during the time intervals $[it_0, (i + 1)t_0]$, $i = 0, 1, \ldots$, are the same functions of t, we say that the schedule is *periodic* with *period* t_0.

Given k agent speeds $v_1, \ldots, v_k > 0$, the goal is to find a schedule for which the idle time is minimum. A straightforward volume argument from [1] yields the lower bound $I \geq 1/\sum_{i=1}^{k} v_i$. This lower bound applies for both the segment and the circle variant of the problem, and for any speed setting.

For the segment variant, Czyzowicz et al. [1] proposed a simple partitioning strategy, algorithm $\mathcal{A}_1$, where each agent moves back and forth in a segment whose length is proportional with its speed. Algorithm $\mathcal{A}_1$ is *universal* in the sense that is applicable for any speed setting $v_1, \ldots, v_k > 0$ for the agents. $\mathcal{A}_1$ has been proved to be optimal for uniform speeds [1], i.e., when all maximum speeds are equal. It has been conjectured [1] that it is optimal for any speed setting, however this was recently disproved by Kawamura and Kobayashi [2] with two examples (periodic schedules) that only barely invalidate the conjecture. It is worth mentioning that the idle time achieved by $\mathcal{A}_1$ is $2/\sum_{i=1}^{k} v_i$ and thereby $\mathcal{A}_1$ yields a 2-approximation algorithm for the shortest idle time. The current best lower bound examples have an idle time of about $0.98 \left(2/\sum_{i=1}^{k} v_i\right)$.

For the circle variant, no universal algorithm has been proposed to be optimal. However, if the maximum speeds of the agents are the same, i.e., $v_1 = \ldots = v_k = v$, then placing the agents uniformly around the circle and moving in the same direction yields the minimum idle time for this setting. Indeed, the idle time is $1/(kv) = 1/\sum_{i=1}^{k} v_i$, matching the lower bound mentioned earlier.

Under the restriction that all agents must move in the same, say clockwise direction, Czyzowicz et al. [1] conjectured that the following algorithm $\mathcal{A}_2$ is optimal: Let $v_1 \geq v_2 \geq \ldots \geq v_k$. Let r be such that $\max_{1 \leq i \leq k} iv_i = rv_r$. Place the agents $a_1, a_2, \ldots, a_r$ at equal distances of $1/r$ around the unit circle, each moving clockwise at the same speed v_r. Discard the remaining agents, if any. Since all agents move in the same direction, we also refer to $\mathcal{A}_2$ as the "runners" algorithm. Observe that $\mathcal{A}_2$ is also universal. Its idle time is $1/\max_{1 \leq i \leq k} iv_i$ [1, Theorem 2]. The conjectured optimality of $\mathcal{A}_2$ is still open.

Notation and terminology. Write $H_n = \sum_{i=1}^{n} 1/i$. A *unit* circle (resp., segment) is one of unit length. For a given patrolling algorithm $\mathcal{A}$, using maximum speeds $v_1, \ldots, v_k > 0$, let $\mathtt{idle}(\mathcal{A}, v_1, \ldots, v_k)$, or just $\mathtt{idle}(\mathcal{A})$ if there is no danger of confusion, denote its idle time.

Given k agents with maximum speeds $v_1, \ldots, v_k > 0$, and a patrolling algorithm $\mathcal{A}$, let $L(\mathcal{A}, v_1, \ldots, v_k)$ denote the maximum length of a segment patrolled

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by these agents using algorithm $\mathcal{A}$. Since the partition-based algorithm was conjectured to be optimal for a segment, it is natural to define the ratio of performance for any other algorithm $\mathcal{A}'$ over the existing partition-based algorithm $\mathcal{A}_1$ as $\rho = \rho(\mathcal{A}', \mathcal{A}_1) = L(\mathcal{A}', v_1, \dots, v_k)/L(\mathcal{A}_1, v_1, \dots, v_k)$, where $L(\mathcal{A}_1, v_1, \dots, v_k) = \left(\sum_{i=1}^k v_i\right)/2$. This ratio can be used to evaluate strategies for patrolling—higher ratio implies better strategy. More generally one can compare two arbitrary strategies $\mathcal{A}', \mathcal{A}''$ via their lengths $L(\mathcal{A}', v_1, \dots, v_k)$ and $L(\mathcal{A}'', v_1, \dots, v_k)$. It is worth to keep in mind the equivalence between comparing different strategies via either their ratio or their idle time: if two algorithms compare with each other with ratio ρ in the length measure, the ratio of their idle times is $1/\rho$.

We use *distance-time diagrams* to plot the agent trajectories with respect to time. The x-coordinate represents distance along the fence and the y-coordinate represents time. For a constant-speed trajectory connecting (x_1, y_1) and (x_2, y_2) in the diagram, construct a shaded parallelogram with vertices, (x_1, y_1), (x_1, y_1+I), (x_2, y_2), $(x_2, y_2 + I)$, where I denotes the idle time (in most of our cases, $I = 1$) and the shaded region represents the covered (guarded) region. A schedule for the agents ensures idle time I if and only if all area of the diagram in the time interval $[I, \infty)$ is covered.

General observations. 1. *Strategy scalability.* Suppose we have a patrolling strategy with k agents for a fence (open or closed) of length l with ratio ρ (relative to the partition strategy). Then, we can *scale* this strategy for every $l' \neq l$ using k agents as follows. Let $l'/l = c$, then $v'_i = cv_i, 1 \leq i \leq k$, where v'_i is the scaled speed of a_i. The waiting times used in the strategy at specific positions for agents need not to be scaled. One can check that the ratio ρ remains unchanged.

2. *Strategy extension.* Suppose we have a patrolling strategy with k agents for a fence (open or closed) of length l with ratio $\rho > 1$ (relative to the partition strategy). Then for any $k' > k$, there exists a a patrolling strategy with k' agents for a fence of length $l' > l$ with ratio $\rho' > 1$: use $m = k' - k$ additional agents with $\sum_{i=k+1}^{k'} v_i = 2(l' - l)$ to patrol $l' - l$ using the partition strategy. Now if $\rho = \frac{a}{b} > 1$, then one can check that $\rho' = \frac{a+2(l'-l)}{b+2(l'-l)} > 1$. It follows from the results of Kawamura and Kobayashi [2] and the above observation that the partition based algorithm is not optimal for a segment for any $k \geq 6$, and k suitable speeds.

Our results.

1. For every integer $x \geq 2$ there exist $k = 4x+1$ agents with $\sum_{i=1}^k v_i = 48x+3$ and a guarding schedule for a segment of length $25x/3$. Alternatively, for every integer $x \geq 2$ there exist $k = 4x + 1$ agents with suitable speeds $v_1, \dots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\frac{48x+3}{50x} \frac{2}{\sum_{i=1}^k v_i}$. In particular, for every $\varepsilon > 0$, there exist k agents with suitable speeds $v_1, \dots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\left(\frac{24}{25} + \varepsilon\right) \frac{2}{\sum_{i=1}^k v_i}$. See Theorem 3, Section 2.
2. For every $k \geq 4$, there exist maximum speeds $v_1 \geq v_2 \geq \ldots \geq v_k$ and a new patrolling algorithm $\mathcal{A}_3$ that yields an idle time better than that achieved by both $\mathcal{A}_1$ and $\mathcal{A}_2$. In particular, for large k, the idle time of $\mathcal{A}_3$ with these speeds is about 2/3 of that achieved by $\mathcal{A}_1$ and $\mathcal{A}_2$. See Proposition 1, Section 3.
3. Consider the unit circle, where all agents are required to move in the same direction. For every $t > 0$, there exists $k = k(t) = O(e^t)$ and a schedule for the system of agents with maximum speeds $v_i = 1/i$, $i = 1, \dots, k$, that ensures an idle time < 1 during the time interval $[0, t]$. See Proposition 2, Section 4.
4. For every $k \geq 2$, there exist maximum speeds $v_1 \geq v_2 \geq \ldots \geq v_k$ so that there exists an optimal schedule (patrolling algorithm) for the circle that does not use up to $k - 1$ of the agents $a_2, \dots, a_k$. In contrast, for a segment, any optimal schedule must use all agents. See Proposition 3, Section 4.
5. There exist settings in which if all k agents are used by a patrolling algorithm, then some agent(s) need overtake (pass) other agent(s). This follows from Proposition 3 and partially answers a question left open by Czyzowicz et al. [1, Section 3].
6. When agents have some radius of visibility, there exists instances in which a zero "speed budget" suffices for guarding. E.g., k stationary agents with radii of visibility $r_1, \dots, r_k$, can guard a segment of length $2\sum_{i=1}^k r_i$. This partially answers another question left open by Czyzowicz et al. [1, Section 3].

2 An improved idle time for open fence patrolling

In the paper by Kawamura and Kobayashi [2], the first example with 6 agents has $\rho = 42/41$ and the second example with 9 agents has $\rho = 100/99$. By repeating the strategy from the second example (with 9 agents) with a larger number of agents we improve the ratio to $25/24 - \varepsilon$ for any $\varepsilon > 0$. We need two technical lemmas.

Lemma 1 *Consider a segment of length $L = \frac{25}{3}$ such that three agents a_1, a_2, a_3 are patrolling perpetually each with speed of 5 and generating an alternating sequence of uncovered triangles $T_2, T_1, T_2, T_1, \dots,$ as shown in the distance-time diagram in Fig. 1. Denote the vertical distances between consecutive occurrences of T_1 and T_2 by δ_{12} and between consecutive occurrences of*

T_2 and T_1 by δ_{21}. Denote the bases of T_1 and T_2 by b_1 and b_2 respectively, and the heights of T_1 and T_2 by h_1 and h_2 respectively . Then

(i) $\frac{10}{3}$ *is a period of the schedule.*

(ii) *T_1 and T_2 are congruent; further, $b_1 = b_2 = \frac{1}{3}$, $\delta_{12} = \delta_{21} = \frac{4}{3}$, and $h_1 = h_2 = \frac{5}{6}$.*

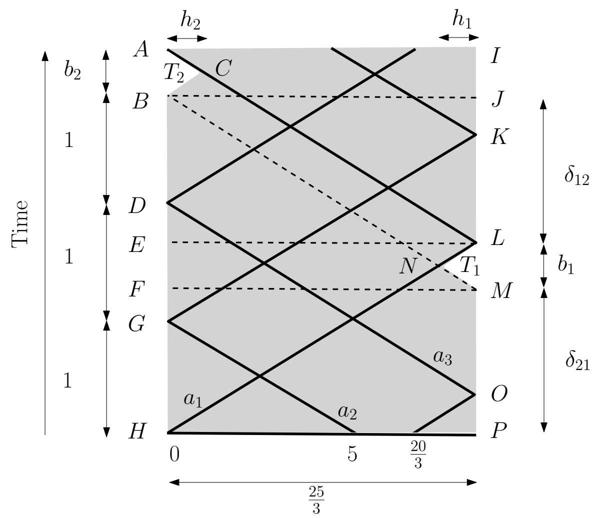

Figure 1: Three agents each with a speed of 5 patrolling a fence of length 25/3; their start positions are 0, 5, and 20/3, respectively. Figure is *not* to scale.

Proof. (i) Observe that a_1, a_2 and a_3 reach the left endpoint of the segment at times $2(25/3)/5 = 10/3$, $5/5 = 1$, and $(25/3 + 5/3)/5 = 2$, respectively. During the time interval $[0, 10/3]$, each agent traverses the distance $2L$ and the positions and directions of the agents at time $t = 10/3$ are the same as those at time $t = 0$. Hence 10/3 is a period for their schedule.

(ii) Since $AL \parallel BM$ and $AB \parallel LM$, we have $b_1 = b_2$. Since L is the midpoint of IP, we have $\delta_{12}+b_2 = \delta_{21}+b_1$, thus $\delta_{12} = \delta_{21}$. Since all the agents have same speed, 5, all the trajectory line segments in the distance-time diagram have the same slope, 1/5. Hence $\angle BAC = \angle ABC = \angle MLN = \angle LMN$. Thus, T_1 is similar to T_2. Since $b_1 = b_2$, T_1 is congruent to T_2, hence $h_1 = h_2$.

Put $b = b_1$, $h = h_1$, and $\delta = \delta_{12}$. Recall from (i) that $|AH| = 10/3$. By construction, we have $|BD| = 1$, thus $|BH| = |BD| + |DG| + |GH| = 1 + 1 + 1 = 3$. We also have $|AH| = b + |BH|$, thus $b = 10/3 - 3 = 1/3$. Since L is the midpoint of IP, we have $\delta + b = 5/3$, thus $\delta = 5/3 - b = 4/3$.

Let $x(N)$ denote the x-coordinate of point N; then $x(N) + h = 25/3$. To compute $x(N)$ we compute the intersection of the two segments HL and BM. We have $H = (0, 0)$, $L = (25/3, 5/3)$, $B = (0, 3)$, and $M = (25/3, 4/3)$. The equations of HL and BM are $HL : x = 5y$ and $BM : x + 5y = 15$, and solving for x yields $x = 15/2$, and consequently $h = 25/3 - 15/2 = 5/6$. □

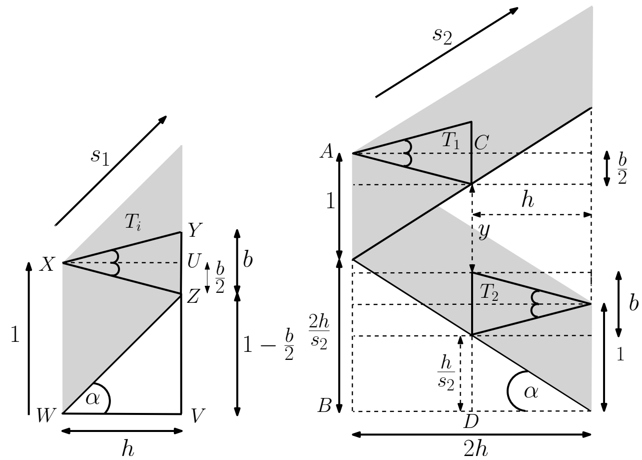

Figure 2: Left: agent covering an uncovered triangle T_i. Right: agent covering an alternate sequence of congruent triangles T_1, T_2, with collinear bases.

Lemma 2 (i) *Let s_1 be the speed of an agent needed to cover an uncovered isosceles triangle T_i; refer to Fig. 2(left). Then $s_1 = \frac{h}{1-b/2}$, where $b < 1$ and h are the base and height of T_i, respectively.*

(ii) *Let s_2 be the speed of an agent needed to cover an alternate sequence of congruent isosceles triangles T_1, T_2 with bases on same vertical line; refer to Fig. 2(right). Then $s_2 = \frac{h}{3b/2+y-1}$ where y is the vertical distance between the triangles, $b < 1$ is the base and h is the height of the congruent triangles.*

Proof. (i) In Fig. 2(left), $\tan\alpha = 1/s_1$, $|UZ| = b/2$, hence $|VZ| = 1 - b/2$. Also, $\frac{|VZ|}{|WV|} = \tan\alpha = \frac{1-b/2}{h} = \frac{1}{s_1}$, which yields $s_1 = \frac{h}{1-b/2}$.

(ii) In Fig. 2(right), $|AB| = 1 + \frac{2h}{s_2}$. Also, $|CD| = \frac{b}{2} + y + b + \frac{h}{s_2}$. Equating $1 + \frac{2h}{s_2} = \frac{3b}{2} + y + \frac{h}{s_2}$ and solving for s_2, we get $s_2 = \frac{h}{3b/2+y-1}$. □

Theorem 3 *For every integer $x \geq 2$, there exist $k = 4x + 1$ agents with $\sum_{i=1}^{k} v_i = 48x + 3$ and a guarding schedule for a segment of length $25x/3$. Alternatively, for every integer $x \geq 2$ there exist $k = 4x + 1$ agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\frac{48x+3}{50x} \frac{2}{\sum_{i=1}^{k} v_i}$. In particular, for every $\varepsilon > 0$, there exist k agents with suitable speeds $v_1, \ldots, v_k$, and a guarding schedule for a unit segment that achieves idle time at most $\left(\frac{24}{25} + \varepsilon\right) \frac{2}{\sum_{i=1}^{k} v_i}$.*

Proof. Refer to Fig. 3. We use a long fence divided into x blocks; each block is of length 25/3. Each block has 3 agents each of speed 5 running in zig-zag fashion.

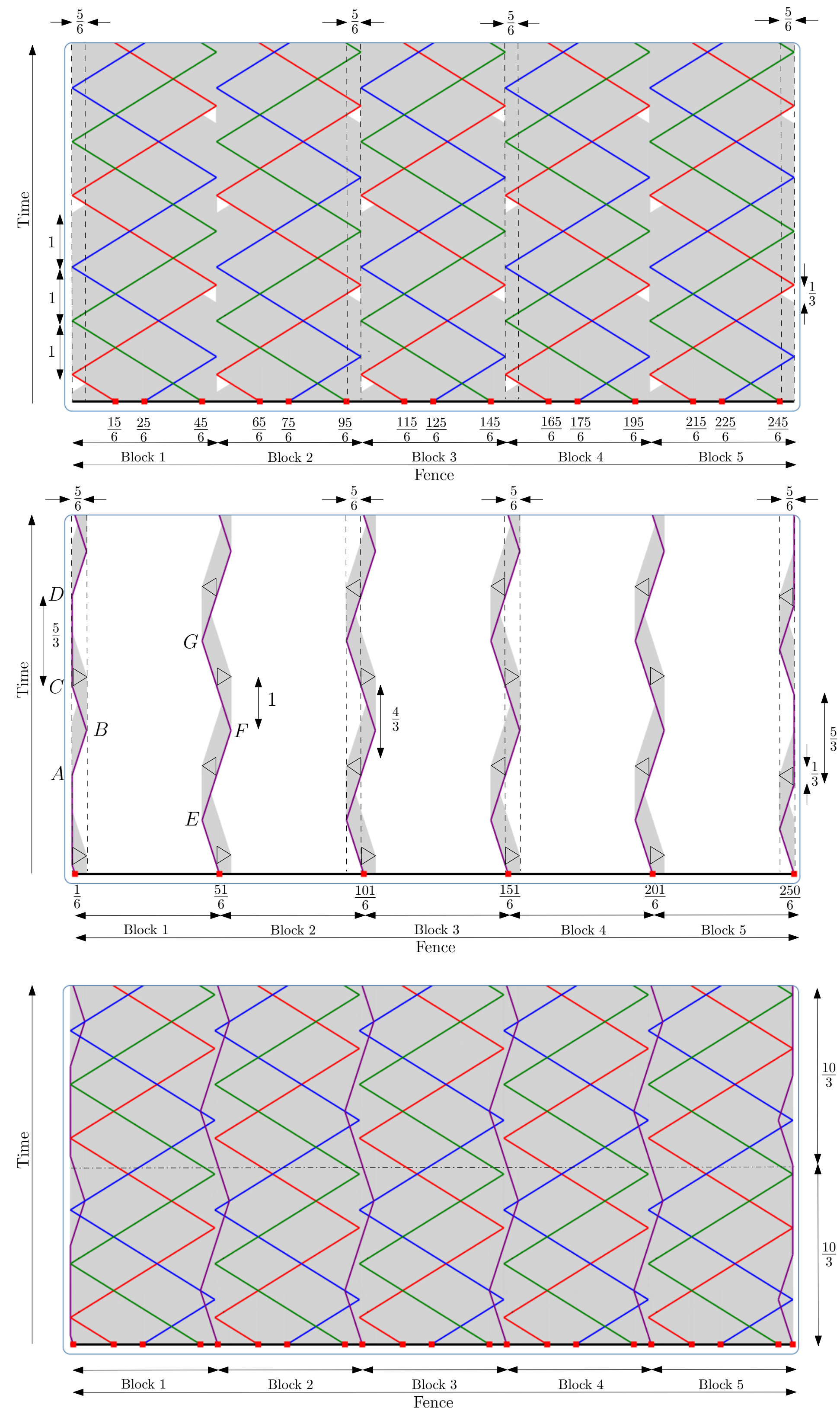

Figure 3: Top: iterative construction with 5 blocks; each block has three agents with speed 5. Middle: six agents with speed 1. Bottom: patrolling strategy for 5 blocks using 21 agents for two time periods (starting at $t = 1/3$ relative to Fig. 1); the block length is $25/3$ and the time period is $10/3$.

Consecutive blocks share one agent of speed 1 which covers the uncovered triangles from the trajectories of the zig-zag agents in the distance-time diagram. The first and the last block use two agents of speed 1 not shared by any other block. The setting of these speeds is explained below.

From Lemma 1(ii), we conclude that all the uncovered triangles generated by the agents of speed 5 are congruent and their base is $b = 1/3$ and their height is $h = 5/6$. By Lemma 2(i), we can set the speeds of the agents not shared by consecutive blocks to $s_1 = \frac{5/6}{1-1/6} = 1$. Also, in our strategy, Lemma 1(ii) yields $y = \delta = 4/3$. Hence, by Lemma 2(ii), we can set the speeds of the agents shared by consecutive blocks to $s_2 = \frac{5/6}{1/2+4/3-1} = 1$.

In our strategy, we have 3 types of agents: agents running with speed 5 as in Fig. 3(top), unit speed agents not shared by 2 consecutive blocks and unit speed agents shared by two consecutive blocks as in Fig. 3(middle). By Lemma 1(i), the agents of first type have period 10/3. In Fig. 3(middle), there are two agents of second type and both have a similar trajectory. Thus, it is enough to verify for the leftmost unit speed agent. It takes 5/6 time from A to B and again 5/6 time from B to C. Next, it waits for 5/3 time at C. Hence after $5/6+5/6+5/3 = 10/3$ time, its position and direction at D is same as that at A. Hence, its time period is 10/3. For the agents of third type, refer to Fig. 3(middle): it takes 10/6 time from E to F and 10/6 time from F to G. Thus, arguing as above, its time period is 10/3. Hence, overall the time period of the strategy is 10/3.

For x blocks, we use $3x + (x + 1) = 4x + 1$ agents. The sum of all speeds is $5(3x) + 1(x + 1) = 16x + 1$ and the total fence length is $\frac{25x}{3}$. The resulting ratio is $\rho = \frac{25x}{3} / \frac{16x+1}{2} = \frac{50x}{48x+3}$. For example, when $x = 2$ we reobtain the bound of Kawamura and Kobayashi [2], when $x = 39, \rho = \frac{104}{100}$ and further on, $\rho \underset{x\to\infty}{\longrightarrow} \frac{25}{24}$. □

3 A new algorithm for closed fence patrolling

Czyzowicz et al. [1, Theorem 5] showed that for $k = 3$ there exist speed settings and an algorithm that achieves an idle time better than both $\mathcal{A}_1$ and $\mathcal{A}_2$ in this case: 35/36 versus 12/11 and 1. We extend this result for any $k \geq 4$.

Proposition 1 *For every $k \geq 4$, there exist maximum speeds $v_1 > v_2 \geq \ldots \geq v_k$ so that a new patrolling algorithm $\mathcal{A}_3$ (we refer to as the "train algorithm") yields an idle time better than that achieved by both $\mathcal{A}_1$ and $\mathcal{A}_2$. In particular, for large k, the idle time of $\mathcal{A}_3$ with these speeds is about $2/3$ of that achieved by $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Proof. We will need $v_1 > v_2$ in this algorithm. Place the $k-1$ agents $a_2, \ldots, a_k$ at equal distances, x on the unit circle, and let them move all clockwise at the same speed v_k; we say that $a_2, \ldots, a_k$ make a "train". Let a_1 move back and forth (i.e., clockwise and counterclockwise) on the moving segment of length $1 - (k-2)x$, i.e., between the start and the end of the train. Refer to Fig. 4. Consider the speed setting: $v_1 = a$,

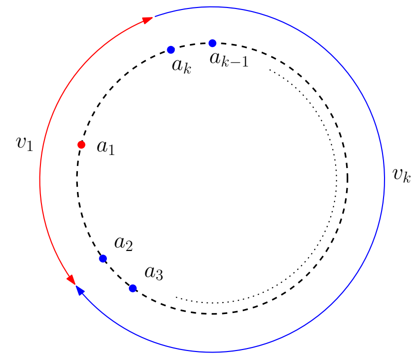

Figure 4: Train algorithm: the train $a_2, \ldots, a_k$ moving unidirectionally with speed v_k and the bidirectional agent a_1 with speed v_1.

$v_2 = \ldots = v_k = b$, where $a > b$, and $\max_{1\leq i\leq k} iv_i = kb$ (i.e., $a \leq kb$). Put $y = 1-(k-2)x$. To determine the idle time, x/b, write: $[1-(k-2)x]\left(\frac{1}{a-b} + \frac{1}{a+b}\right) = \frac{x}{b}$, or equivalently, $\frac{2ay}{a^2-b^2} = \frac{1-y}{(k-2)b}$. Solving for x/b yields

$$\texttt{idle}(\mathcal{A}_3) = \frac{2a}{a^2 - b^2 + 2(k-2)ab}.$$

For our setting, we also have

$$\texttt{idle}(\mathcal{A}_1) = \frac{2}{a + (k-1)b}, \text{ and } \texttt{idle}(\mathcal{A}_2) = \frac{1}{kb}.$$

Write $t = a/b$. It can be checked that for $k \geq 4$, $\texttt{idle}(\mathcal{A}_3) \leq \texttt{idle}(\mathcal{A}_1)$ and $\texttt{idle}(\mathcal{A}_3) \leq \texttt{idle}(\mathcal{A}_2)$ when $a^2 - b^2 - 4ab \geq 0$, i.e., $t \geq 2 + \sqrt{5}$. In particular, for $a = 1$, and $b = 1/k$ (note that $a \leq kb$), we have

$$\texttt{idle}(\mathcal{A}_3) = \frac{2}{1 - 1/k^2 + 2(k-2)/k} \underset{k\to\infty}{\longrightarrow} \frac{2}{3},$$

while $\texttt{idle}(\mathcal{A}_1) = \frac{2}{1+(k-1)/k} \underset{k\to\infty}{\longrightarrow} 1$ and $\texttt{idle}(\mathcal{A}_2) = \frac{1}{k(1/k)} = 1$. □

4 Remarks

Finite time circle patrolling. While we cannot confirm the conjectured optimality of $\mathcal{A}_2$—in particular, for the system of agents with maximum speeds $v_i = 1/i$, $i = 1, \ldots, k$, acting on the unit circle, we would have $\texttt{idle}(\mathcal{A}_2) = 1$—we can achieve an idle time below 1 in this setting for an arbitrarily long time, provided we choose k large enough. Obviously for this setting we have $\texttt{idle}(\mathcal{A}_2) \leq 1$, which is already achieved by the agent a_1 with the highest (here unit) speed, and the conjecture says that $\texttt{idle}(\mathcal{A}_2) < 1$ does not hold.

Proposition 2 *Consider the unit circle, where all agents are required to move in the same direction. For every* $t > 0$*, there exists* $k = k(t) = O(e^t)$ *and a schedule for the system of agents with maximum speeds* $v_i = 1/i$, $i = 1, \ldots, k$*, that ensures an idle time* < 1 *during the time interval* $[0, t]$.

Proof. We construct a schedule with an idle time smaller than 1. Let $a_1(t) = t \mod 1$ denote the position of agent a_1 at time t; in particular $a_1(0) = 0$ with a_1 moving clockwise at maximum (unit) speed. We ensure that for each $t \geq 1$ there exists an agent that covers the interval $[t - \delta_1, t + \delta_2]$, for suitable $\delta_1, \delta_2 > 0$ before a_1 reaches this interval at time $t - \delta_1$. (We ignore any other contribution of this agent to the overall coverage.) We use many different agents to cover all time instances $t' \in [1, t]$. To this end we use the well-known fact that the harmonic series $\sum_1^\infty 1/i$ is divergent, more precisely that $H_k \geq \ln(k + 1)$.

To start with, put $u_1 = 1$ as the first uncovered time instant t', and $i = 2$ as the index of the next unused agent. Having defined u_{i-1}, initiate the movement of the next agent a_i at time $u_{i-1} - 1/2$ from the position $u_{i-1} - 1/(8i)$. Its speed is $1/i$ and during a time interval of $1/2$, the agent will traverse a distance equal to $1/(2i)$. Hence the agent's position at time u_{i-1} will be $u_{i-1} - 1/(8i) + 1/(2i) = u_{i-1} + 3/(8i)$. Now set $u_i = u_{i-1} + 3/(8i)$. In particular, $u_2 = 1 + 3/(8 \cdot 2)$ is the second uncovered time (to be covered by another agent), and $u_3 = 1 + 3/(8 \cdot 2) + 3/(8 \cdot 3)$ is the next such term. The solution of the recurrence is $u_k = \frac{5}{8} + \frac{3}{8} H_k$, and we need $u_k \geq t$. Since $H_k \geq \ln(k + 1)$, it follows that $k = O(e^t)$ agents suffice to cover the time interval $[0, t]$ and ensure an idle time smaller than 1 in this way. □

Useless agents for circle patrolling. Czyzowicz et al. [1] showed that for $k = 2$ there exist speed settings when an optimal schedule does not use one of the agents. Here we extend this result for all $k \geq 2$:

Proposition 3 (i) *For every* $k \geq 2$*, there exist maximum speeds* $v_1 \geq v_2 \geq \ldots \geq v_k > 0$ *and an optimal schedule (patrolling algorithm) for the circle with these speeds that does not use up to* $k - 1$ *of the agents* $a_2, \ldots, a_k$. (ii) *In contrast, for a segment, any optimal schedule must use all agents.*

Proof. (i) Let $v_1 = 1$ and $v_2 = \ldots = v_k = \varepsilon/k$, for a small positive $\varepsilon \leq 1/300$, and C be a unit length circle. Obviously by using agent a_1 alone (moving perpetually clockwise) we can achieve unit idle time. Assume for contradiction that there exists a schedule achieving an idle time less than 1. Let $a_1(t) = t \mod 1$ denote the position of agent a_1 at time t. Assume without loss of generality that $a_1(0) = 0$ and consider the time interval $[0, 2]$. For $2 \leq i \leq k$, let J_i be the interval of points visited by agent a_i during the time interval $[0, 2]$, and put $J = \cup_{i=2}^k J_i$. We have $|J_i| \leq 2\varepsilon/k$, thus $|J| \leq 2\varepsilon$. We make the following observations:

1. $a_1(1) \in [-2\varepsilon, 2\varepsilon]$. Indeed, if $a_1(1) \notin [-2\varepsilon, 2\varepsilon]$, then either some point in $[-2\varepsilon, 2\varepsilon]$ is not visited by any agent during the time interval $[0, 1]$, or some point in $C \setminus [-2\varepsilon, 2\varepsilon]$ is not visited by any agent during the time interval $[0, 1]$.
2. a_1 has done almost a complete (say, clockwise) rotation along C during the time interval $[0, 1]$, i.e., it starts at $0 \in [-2\varepsilon, 2\varepsilon]$ and ends in $[-2\varepsilon, 2\varepsilon]$, otherwise some point in $C \setminus [-2\varepsilon, 2\varepsilon]$ is not visited during the time interval $[0, 1]$.
3. $a_1(2) \in [-4\varepsilon, 4\varepsilon]$, by a similar argument.
4. a_1 has done almost a complete rotation along C during the time interval $[1, 2]$, i.e., it starts in $[-2\varepsilon, 2\varepsilon]$ and ends in $[-4\varepsilon, 4\varepsilon]$. Moreover this rotation must be in the same clockwise sense as the previous one, since otherwise there would exist points not visited for at least one unit of time.

Pick three points $x_1, x_2, x_3 \in C \setminus J$ close to $1/4$, $2/4$, and $3/4$, respectively, i.e., $|x_i - i/4| \leq 1/100$, for $i = 1, 2, 3$. By Observations 2 and 4, these three points must be visited by a_1 in the first two rotations during the time interval $[0, 2]$ in the order $x_1, x_2, x_3, x_1, x_2, x_3$. Since a_1 has unit speed, successive visits to x_1 are at least one time unit apart, contradicting the assumption that the idle time of the schedule is less than 1.

(ii) Given $v_1 \geq v_2 \geq \ldots \geq v_k > 0$, assume for contradiction that there is an optimal guarding schedule with unit idle time for a segment s of maximum length that does not use agent a_j (with maximum speed v_j), for some $1 \leq j \leq k$. Extend s at one end by a subsegment of length $v_j/2$ and assign a_j to this subsegment to move back and forth from one end to the other, perpetually. We now have a guarding schedule with unit idle time for a segment longer than s, which is a contradiction. □

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