Stabbing Polygonal Chains with Rays is Hard to Approximate

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Abstract

We study a geometric hitting set problem involving unirectional rays and curves in the plane.

We show that this problem is hard to approximate within a logarithmic factor even when the curves are convex polygonal x-monotone chains. Additionally, it is hard to approximate within a factor of $\frac{7}{6}$ even when the curves are line segments with bounded slopes. Lastly, we demonstrate that the problem is $W[2]$-complete when the curves are convex polygonal x-monotone chains and is $W[1]$-hard when the curves are line segments.

1 Introduction

Motivated by art-gallery problems such as terrain-guarding and minimum-link watchman route, Katz, Mitchell and Nir [7] studied a family of geometric stabbing/hitting problems involving orthogonal line segments and rays in the plane. Among other problems, they introduced the following problem of *Stabbing Segments with Rays* (SSR).

PROBLEM: SSR.

INPUT: A set S of non-vertical line segments and a set R of upwards rays.

OUTPUT: A minimum cardinality subset R' of R so that for each $s \in S$ there exists $r \in R'$ for which $r \cap s \neq \emptyset$.

In addition to the SSR problem we also consider the more general problem of *Stabbing polygonal Chains with Rays* (SCR); defined below.

PROBLEM: SCR.

INPUT: A set C of polygonal chains and a set R of upwards rays.

OUTPUT: A minimum cardinality subset R' of R so that for each $c \in C$ there exists $r \in R'$ for which $r \cap c \neq \emptyset$.

Katz, Mitchell and Nir [7] presented an exact poly-time solution for the variant of SSR in which the line segments in S are non-intersecting, and left the more general problem (involving intersecting line segments) open.

An equivalent problem was studied by Chan and Grant [1] who refer to it as the problem of *hitting-set of downward shadows of line segments in the plane with points*. A *downward shadow* of a set $Y \subset \mathbb{R}^2$ is the set of all points $p = (p_x, p_y)$ in the plane for which there exists a point $q = (q_x, q_y)$ in Y with $q_x = p_x$ and $q_y \geq p_y$. Chan and Grant have also presented an exact poly-time solution; both solutions [1, 7] are similar and based on dynamic programming.

Chan and Grant [1] studied several other related problems, including covering and packing. For example they showed that the hitting set of downward shadows of horizontal line segments by points and its "almost-dual" problem (covering points with downward shadows of horizontal line segments) are both poly-time solvable. They further showed that the covering of points with downward shadows of 2-intersecting[1] x-monotone curves is poly-time solvable.

In contrast to the fact that both covering and hitting problems with downward shadows of horizontal line segments are poly-time solvable, for downward shadows of 2-intersecting x-monotone curves, things are different. Chan and Grant [1] remarked that a naive attempts to generalize their solution to a solution for the hitting problem of downward shadows of curves appear to fail. They left it as an open problem to determine whether the hitting set problem involving downward shadows of 2-intersecting x-monotone curves in the plane is APX-hard, has a PTAS, or perhaps is even poly-time solvable. In this paper we show that it is APX-hard, even for the simple case of downward shadows of line segments with bounded slopes.

In Section 2 we show that SCR is hard to approximate within a logarithmic factor. In Section 3 we show that although simpler, SSR is still APX-hard and is hard to approximate within a factor of $\frac{7}{6}$. We further observe that these results hold even if the slopes of all line segments are bounded. In Section 4 we show that SCR is

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[1] A collection C of curves is said to be *k-intersecting* if every two curves in C intersect at most k times.

$W[2]$-complete and that SSR is $W[1]$-hard; thus both are most unlikely to be fixed-parameter intractable.

2 Stabbing polygonal chains with rays

We show below that SCR is APX-hard via a reduction from Hitting-Set (HS).

PROBLEM: HS.

INPUT: A set U and a collection $\mathcal{S}$ of subsets of U.

OUTPUT: A minimum cardinality subset U' of U so that for every $S \in \mathcal{S}$ we have that $S \cap U' \neq \emptyset$.

Theorem 1 *SCR is APX-hard, and is hard to approximate within a logarithmic factor.*

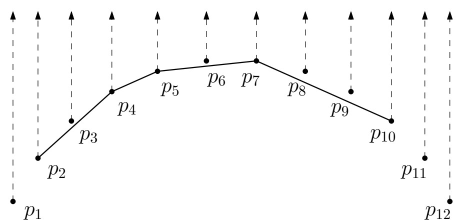

Figure 1: A chain corresponding to $\{2, 4, 5, 7, 10\}$. It can be stabbed only by the rays with origins at the corresponding points.

Proof. We show that HS is reducible to SCR via a cost-preserving reduction; since HS is hard to approximate within a logarithmic factor [4], the desired result follows.

Let $\mathcal{S}$ be a collection of subsets of $[n]$. We construct sets C of polygonal chains and R of upwards rays so that a minimum hitting set for $\mathcal{S}$ corresponds to a minimum subset of R that stabs all chains in C and vice versa. Indeed, let $P = \{p_i : 1 \leq i \leq n\}$ be a point set lying on the upper half of a circle, left to right, and let R be the set of upwards rays with origins in P. Starting with an empty set C, for each set $S \in \mathcal{S}$, add to C the (convex) polygonal chain with vertices at points corresponding to elements of S. More precisely, put $S = \{i_1, i_2, \ldots, i_m\} \subseteq [n]$ where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ then the corresponding chain is $c = \langle p_{i_1}, p_{i_2}, \ldots, p_{i_m} \rangle$; see Figure 1. It is easy to see that a chain $c \in C$ is stabbed by exactly those rays with origins at its vertices, just as the set $S \in \mathcal{S}$ is hit by exactly its elements. That is, a minimum hitting set for $\mathcal{S}$ corresponds to a minimum subset of R that stabs all chains in C (of the same cardinality) and vice versa. □

3 Stabbing segments

SSR is a special case of SCR. Although simpler, it is still APX-hard. The reduction in the proof of Theorem 1 is applicable also for special case in which the collection $\mathcal{S}$ is a collection of pairs, and consequently, the resulting chains are line segments. That is, the reduction in the proof of Theorem 1 is also a reduction from Vertex-Cover (VC) to SSR, as is described in the proof of Theorem 2 below.

Theorem 2 *SSR is APX-hard and is hard to approximate within a factor of* $\frac{7}{6}$.

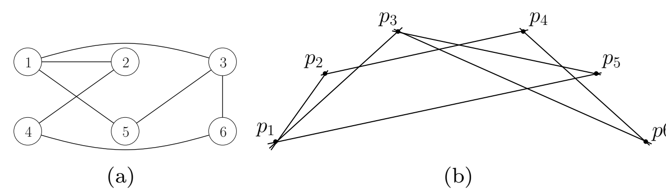

Figure 2: (a) A graph G and (b) its corresponding appearance of SSR.

Proof. Let $G = (V, E)$ be a graph. As in the proof of Theorem 1, P is a point set embedded to the upper half of a circle, now corresponding to the vertices of G, R is a set of upwards rays with origins in P, and S is a set of line segments with endpoints in P where for each edge $(v_i, v_j) \in E$, the line segment $\overline{p_i p_j}$ is in S; see Figure 2 for an illustration. It is easy to see that a minimum vertex cover of G corresponds to a minimum subset of R that stabs all segments in S (of the same cardinality) and vice versa. Since VC is hard to approximate within a factor of $\frac{7}{6}$ [6], then we are done. □

Notice that the reduction in proof of Theorems 1 and 2 could be adjusted, by sliding the points of P, to fit restricted versions of SCR and SSR in which all slopes of the segmental links are within a fixed range. Corollary 3 below follows. We first define $SCR(\alpha, \beta)$ and $SSR(\alpha, \beta)$.

PROBLEM: $SCR(\alpha, \beta)$.

INPUT: Two parameters $0 \leq \alpha < \beta \leq \pi$, a set C of polygonal chains whose segmental links have slopes between α and β, and a set R of upwards rays.

OUTPUT: A minimum cardinality subset R' of R so that for each $c \in C$ there exists $r \in R'$ for which $r \cap c \neq \emptyset$.

PROBLEM: $SSR(\alpha, \beta)$.

INPUT: Two parameters $0 \leq \alpha < \beta \leq \pi$, a set S of line segments with slopes between α and β, and a set R of upwards rays.

OUTPUT: A minimum cardinality subset R' of R so that for each $s \in S$ there exists $r \in R'$ for which $r \cap s \neq \emptyset$.

Corollary 3 *For any* $0 \leq \alpha < \beta \leq \pi$, $SCR(\alpha, \beta)$ *and* $SSR(\alpha, \beta)$ *are APX-hard.*

4 Fixed parameter intractability

The concept of parameterized complexity was introduced by Downey and Fellows [3] (see also [5, 8]). An instance of a parameterized problem is a pair (I, k), where k is a parameter; the complexity of the problem is measured not only with respect to the input size, but also with respect to the parameter k. A parameterized problem is said to be *fixed-parameter tractable* (FPT) with respect to the parameter k if there exists an algorithm for the problem with time complexity $f(k) \cdot p(|I|)$ for some computeable function f and some polynomial p. Downey and Fellows [3] also introduced a theory of *parameterized intractability*; i.e., a hierarchy of complexity classes called the *W-hierarchy.* Some of the classes in this hierarchy are interrelated as follows: $FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq W[3] \cdots$. Generally speaking, a parameterized problem with parameter k is in the class $W[i]$ if it is reducible to a circuit of height i or less, which assigns 1 to at most k inputs. It is most-likely that all above inclusion are strict, namely, there are problems in $W[1]$ that are most likely fixed-parameter intractable and, in particular, that $W[1]$-hard problems are fixed-parameter intractable [3, 5, 8].

In this section we observe that SCR is $W[2]$-complete and demonstrate that SSR is $W[1]$-hard.

The reduction in the proof of Theorem 1 shows that SCR is exactly the same as Hitting Set. Thus SCR is $W[2]$-complete since Hitting Set is $W[2]$-complete [3, 5]. Unfortunately, this is not necessarily true for SSR as the reduction in this case is from 2-Hitting-Set. In general, d-Hitting-Set (i.e., when the size of each subset is at most a constant d) is FPT. Therefore the reduction from Vertex-Cover does not provide any $W[k]$-hardness. We show that SSR is $W[1]$-hard via a reduction from the problem of *stabbing 2-intervals*[2] *with points* (S2I) (defined below) which is known to be $W[1]$-hard [2] (note: S2I is equivalent to the 2-C1P-Set-Cover problem in [2]). We consider all problems to be parameterized with respect to the size of the solution.

PROBLEM: S2I.

INPUT: A collection $2I$ of 2-intervals and a set P of points on the line.

OUTPUT: A minimum cardinality subset P' of P so that for each 2-interval $I \in 2I$, $I = (I_1, I_2)$ there exists a point $p \in P'$ for which $p \in I_1$ or $p \in I_2$.

Theorem 4 *SSR is $W[1]$-hard.*

Proof. We show S2I is reducible to SSR with similar parameters; since the former is $W[1]$-hard [2], the desired result follows. Let $2I$ be a collection of 2-intervals, and P a set of points on the line. We construct sets S of line segments and R of upwards rays so that a minimum subset P' of P that collectively stabs all 2-intervals in $2I$ corresponds to a minimum subset of R that stabs all segments in S and vice versa.

Notice that we may assume that the endpoints of intervals in $2I$ as well as the points in P are integers. Moreover, we may also assume that the endpoints of all intervals in $2I$ are taken from P (otherwise an interval with an endpoint which is not in P can be shortened). We embed the points in P on the upper half of a circle in the same left to right order in which they appear on the line. Starting with an empty set S, for each 2-interval $I = ([i, j], [k, l])$ with $i \leq j \leq k \leq l$, add $s(I)$ to S where $s(I)$ denotes the line segment through the points p_j and p_k whose left and right endpoints are, respectively, above p_i and p_l. Finally, let R be the set of upward rays with origins at the points of P; see Figure 3 for an illustration. It is easy to see that the line segment $s(I)$ where $I = ([i, j], [k, l])$ lies above exactly those points of P that lie in the union of its two intervals. Thus, the rays which can be chosen to stab I are precisely the ones corresponding to points that belong to I. That is, a minimum subset of P that stabs $2I$ corresponds to a minimum subset of R that stabs S (of same cardinality) and vice versa. □

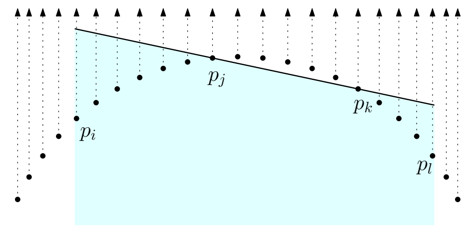

Figure 3: A segment corresponding to a 2-interval $([i, j], [k, l])$. The segment lies above exactly those points that are contained in one of its two intervals.

5 Concluding Remarks

The paper shows that SCR is hard to approximate within a logarithmic factor while SSR is hard to approximate within a constant factor. A first natural open question we raise is either to strengthen the hardness of approximation or to present an approximation algorithm better than the standard $O(\log n)$-approximation algorithm for Hitting-Set. In particular, we hope that the geometric structure that we have observed may be useful for this purpose.

We showed that SCR is $W[2]$-complete and that SSR is $W[1]$-hard, However, it is still open whether SSR is in $W[1]$, $W[2]$-complete, or in $W[2]$ but not $W[2]$-complete.

[2] A 2-interval is a pair of intervals on the line.

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