Planar Graphs with Many Perfect Matchings and Forests*

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Abstract

We determine the number of perfect matchings and forests in a family $T_{r,3}$ of triangulated prism graphs. These results show that the extremal number of perfect matchings in a planar graph with n vertices is bounded below by $\Omega\left(\sqrt[6]{7+\sqrt{37}}^n\right) = \Omega(1.535^n)$ and the extremal number of forests contained in a planar graph with n vertices is bounded below by $\Omega\left(\sqrt[3]{138.74876}^n\right) = \Omega(5.1769^n)$. The current best known upper bounds for these problems are $O(1.565^n)$ and $n \cdot 6.4948^n$, respectively.

1 Introduction

In recent years, numerous papers have investigated the number of special subgraphs that can be found in an arbitrary plane point set or planar graph. Specifically of interest have been triangulations, spanning cycles, spanning trees, forests and perfect matchings [1, 7, 8, 9].

In particular, we will consider the numbers of perfect matchings and forests contained in planar graphs. Given a graph G with $2k$ vertices, a *perfect matching* in G is a subset of k edges of G such that every vertex of G is incident on exactly one edge in the subset. A *forest* in G is any cycle-free spanning subgraph of G.

In this paper we construct a family of plane graphs with many perfect matchings and forests. The family consists of modified *prism graphs*. Let P_r denote the r-vertex path graph, and C_s denote the s-vertex cycle graph. Then we define $P_{r,s} = P_r \times C_s$ to be the r by s prism graph, where $\times$ is the Cartesian graph product. $P_{r,s}$ can be viewed as regular grid on a cylinder, and we create the triangulated prism graph $T_{r,s}$ by systematically adding parallel diagonals to each square of the grid. We additionally modify the graph, if rs is odd, in order to enable the graphs to admit perfect matchings by adding an additional 'center' vertex, connected to one of the boundary copies of C_r. See Figure 1 for an example with rs even and Figure 2 with rs odd.

To facilitate the description of the graphs in some of the following arguments, in planar drawings of triangulated prism graphs of the type seen in Figures 1 and 2, we say that the vertices incident to the unbounded face are on the *outer layer*, and the vertices adjacent to vertices on the outer layer are on the *next layer*.

These graphs are particularly useful in problems of extremal combinatorics on planar graphs, and their variants have appeared in several papers as examples of lower bounds for counting structures in planar graphs. In [6] they were used to compute lower bounds on the number of spanning trees in planar graphs and were called prisms, and in [3] they were used to compute lower bounds on the number of simple cycles and spanning cycles in planar graphs and were called twisted cylinders. Recently, they were used to show lower bounds on the number of directed paths in directed plane straight-line graphs [5]. We attempt to standardize the terminology with respect to the well-known prism graphs, and use a modification of the notation in [6], and call the graphs 'triangulated prism graphs'. We will mainly be interested in the triangle family of graphs $T_{r,3}$. See Figure 2, for an example of $T_{3,3}$.

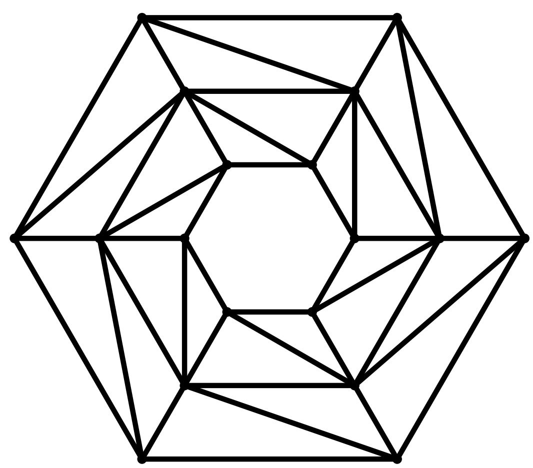

Figure 1: A $T_{3,6}$ Triangulated Prism Graph

First, we consider the problem of counting the number of perfect matchings in triangulated prism graphs. We find an exact recursion for the number of perfect matchings and show that the number of perfect matchings in $T_{r,3}$ grows asymptotically like $\sqrt{7+\sqrt{37}}^r$, where the number of vertices $n \approx 3r$. This implies that the number of perfect matchings in $T_{r,3}$ is $\Omega\left(\sqrt[6]{7+\sqrt{37}}^n\right) = \Omega(1.535^n)$, very close to the known upper bound of

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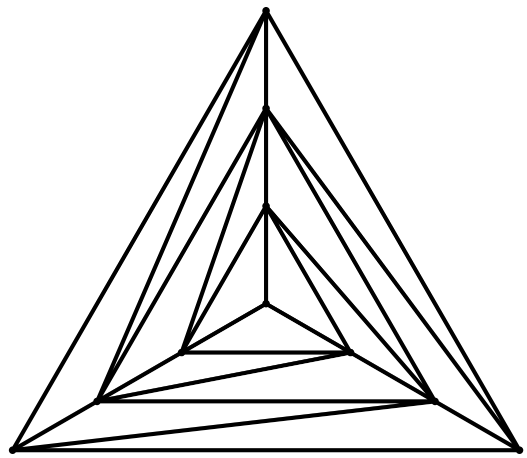

Figure 2: A $T_{3,3}$ Triangulated Prism Graph

$O\left(\sqrt[4]{6}^n\right) = O\left(1.565^n\right)$ from [3]. In addition, numerical evidence suggests that asymptotically this may be the largest number of perfect matchings for triangulated prism graphs with fixed n vertices. The asymptotic number of perfect matchings in $T_{r,6}$ was also computed and was found to exactly match the asymptotic growth rate $\Omega\left(\sqrt{7+\sqrt{37}}^n\right)$ for $T_{r,3}$. This is in contrast with the results for the transfer matrix approaches in [3], where the asymptotic growth rate continued to increase with increasing s until the computations became infeasible.

Secondly, we examine the problem of determining the number of forests in triangulated prism graphs. We find an exact recursion for the number of forests, and show that the number of forests in $T_{r,3}$ grows asymptotically like $\Omega\left(\sqrt[3]{138.749}^n\right)$. This implies that the number of forests in $T_{r,3}$ is $\Omega\left(5.177^n\right)$, while the best known upper bound is $n \cdot 6.4948^n$ from [4].

Lower bounds on these two problems were previously implied by the papers counting similar structures in the same family of graphs. In particular, a lower bound of $\Omega\left(2.0845^n\right)$ spanning cycles in the family of triangulated prism graphs [3] gives a lower bound of $\Omega\left(\sqrt{2.0845^n}\right) = \Omega\left(1.44^n\right)$ perfect matchings, by decomposing spanning cycles into pairs of perfect matchings. Similarly, a lower bound of $\Omega\left(5.0295^n\right)$ is achieved for the number of spanning trees in a triangulated prism graph [6], so that is also a lower bound for the number of forests.

2 Perfect Matchings

In this section, we analyze the number of perfect matchings admitted by triangulated prism graphs $T_{r,3}$ (see Figure 2). Define $\mathrm{pm}(r,s)$ to be the number of perfect matchings in the $T_{r,s}$ graph.

Theorem 1 *The number of perfect matchings in $T_{r,3}$ is $pm(r,3) = \Omega\left(\sqrt{7+\sqrt{37}}^r\right)$.*

Proof. We define a recursion for $\mathrm{pm}\,(r,3)$ based on how many of the outer edges are in the matching, with the base cases corresponding to the matchings in $T_{1,3}$, or $T_{2,3}$, graphs, depending on parity. In the analysis, it is useful to consider the situation of a $T_{r,3}$ graph with two vertices of the outer layer removed, along with their associated edges. Call these the *reduced triangulated prism graphs* $T^-_{r,3}$, and denote the number of perfect matchings in $T^-_{r,3}$ as $\mathrm{pm}^-\,(r,3)$. See Figure 3 for an example of $T^-_{3,3}$.

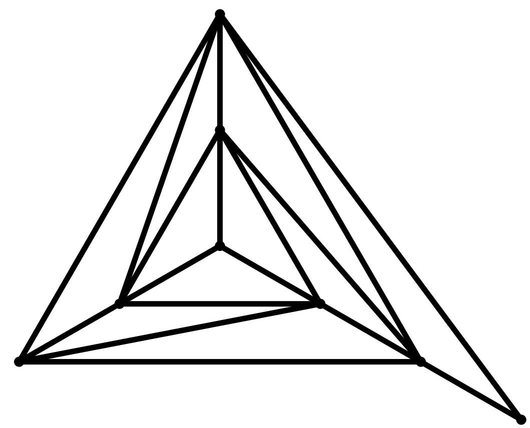

Figure 3: A Reduced $T^-_{3,3}$ Triangulated Prism Graph

We condition on how many of the outermost edges of $T_{r,3}$ are included in the matching, which will either be 0 or 1.

Case 1: If no edges of the outer layer are used in the matching, each vertex on the outer layer is matched to a vertex on the next layer of the graph. There are exactly two choices in matching vertices on the outer layer to vertices on the next layer. In both, after removing the matched vertices and their edges, we are left with a $T_{r-2,3}$ graph. See Figure 4.

Case 2: If an edge of the outer layer is used in the matching, there are three options for which edge to use, and then we have a $T^-_{r,3}$ graph. See Figure 5.

After choosing one of the two symmetric choices for matching the remaining outer vertex of $T^-_{r,3}$, there are two types of graph remaining. In the first case, we have reduced to a $T_{r-2,3}$ graph, while in the second, we have reduced to a $T^-_{r-2,3}$ graph. To determine $\mathrm{pm}^-\,(r,3)$, note that there are two choices for which outer vertex the remaining outer vertex is matched to, and we again condition into two cases: if the outer layer of the reduced prism graph uses one edge or no edges in the perfect matching. In the first case, we have reduced to a single $\mathrm{pm}\,(r-2,3)$. In the second case, there are

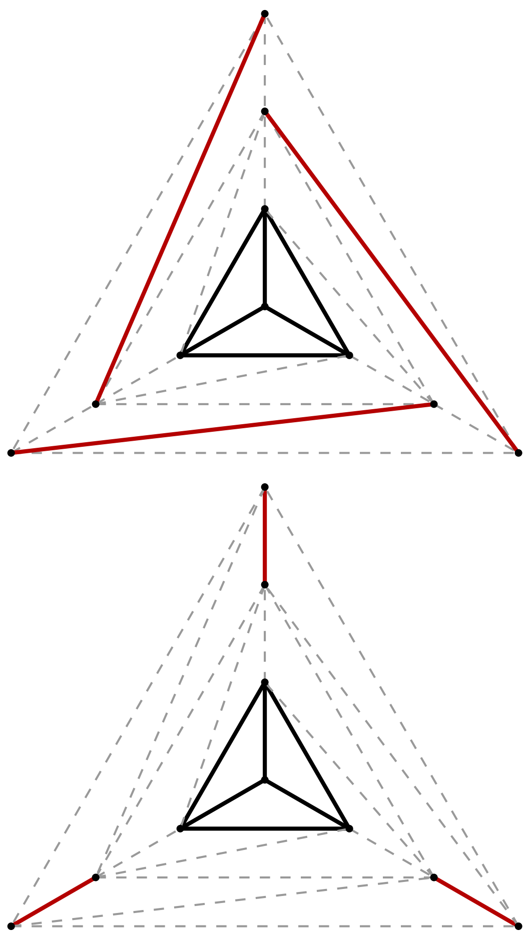

Figure 4: Case 1: If no outer edges are used, there are two possible matchings of the outer vertices. each leaving a $T_{r-2,3}$ graph.

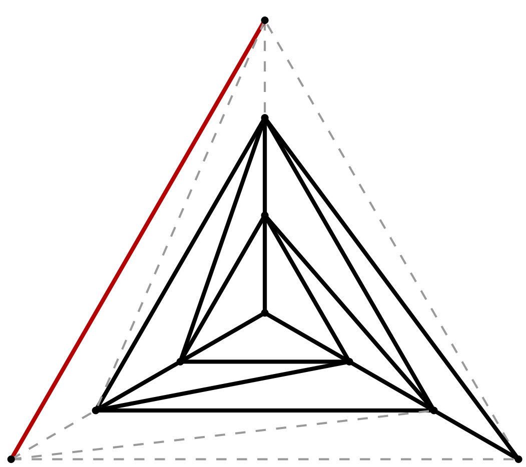

Figure 5: Case 2: If an outer edge is used, we have an reduced prism graph, $T^{-}_{r,3}$.

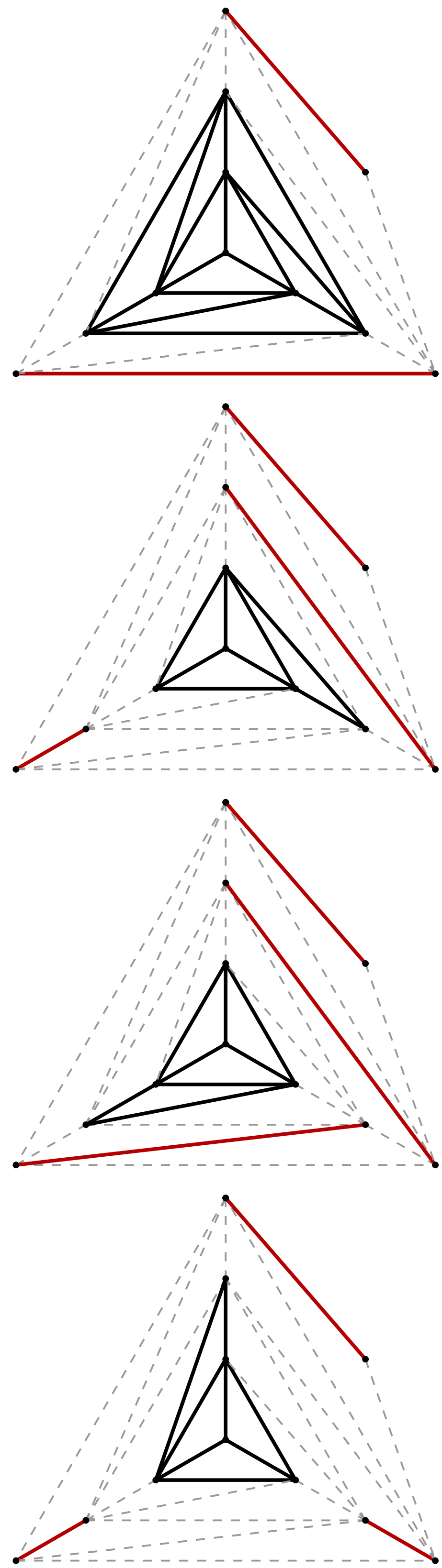

Figure 6: Case 2: For reduced triangulated prism graphs, there are four possibilities for each choice of match of the remaining outer vertex.

three possibilities, each reducing to $\text{pm}^{-}(r-2,3)$. See Figure 6.

Using the above discussion we form the two linked recursions:

$$\text{pm}(r,3) = 2\text{pm}(r-2,3) + 3\text{pm}^{-}(r,3)$$
$$\text{pm}^{-}(r,3) = 2\text{pm}(r-2,3) + 6\text{pm}^{-}(r-2,3)$$

We then have that these two recursive formulas together simplify to:

$$\text{pm}(r,3) = 14\text{pm}(r-2,3) - 12\text{pm}(r-4,3)$$

This linear recursion can be solved exactly, and has asymptotic growth rate on the order of the largest root of $r^4 = 14r^2 - 12$, which is $\sqrt{7+\sqrt{37}}$. Therefore $\text{pm}(r,3) = \Omega\left(\sqrt{7+\sqrt{37}}^{\,r}\right)$. □

Since $r \approx \frac{n}{3}$, we have that $\text{pm}(r,3) = \Omega\left(\sqrt[6]{7+\sqrt{37}}^{\,n}\right) = \Omega(1.535^n)$.

Corollary 2 *There exist n-vertex planar graphs with $\Omega\left(\sqrt[6]{7+\sqrt{37}}^{\,n}\right) = \Omega(1.535^n)$ perfect matchings.*

Note that the best known upper bound for $\text{pm}(G)$ for any planar graph G is derived from Kasteleyn's determinant approach and yields $O\left(\sqrt[4]{6}^{\,n}\right) = O(1.565^n)$ [3], which is extremely close to our bound of $\Omega(1.535^n)$. Computational evidence for the number of perfect matchings in $T_{r,s}$ graphs for $s \neq 3$ indicates that this is the highest asymptotic growth rate of the number of perfect matchings in triangulated prism graphs, although no proof has been found. In particular, a similar analysis for $s = 6$ was done, and the recursions for $\text{pm}(r,6)$ has exactly the same (per vertex) asymptotic growth rate $\Omega\left(\sqrt[6]{7+\sqrt{37}}^{\,n}\right)$ as the $s = 3$ example.

3 Forests

In this section, we analyze the number of forests admitted by triangulated prism graphs $T_{r,3}$. Define $\text{f}(r,s)$ to be the number of forests in the $T_{r,s}$ graph.

Theorem 3 *The number of forests contained in $T_{r,3}$ is $f(r,3) = \Omega(138.749^r)$.*

Proof. We consider cases, conditioned on the decomposition of the outer vertices of $T_{r,3}$ which are in the same connected component with respect to the final forest. In particular, take a standard drawing of $T_{r,3}$ (as in Figure 2) and label the vertices in each layer a, b, and c, clockwise from the topmost vertex in the layer, and define:

- $\text{f}_1(r)$ to be the number of forests in $T_{r,3}$ such that, in the outer layer, a, b, and c are all in separate connected components.
- $\text{f}_2(r)$ to be the number of forests in $T_{r,3}$ such that, in the outer layer, a and b are in the same connected component, and are separated from c.
- $\text{f}_3(r)$ to be the number of forests in $T_{r,3}$ such that, in the outer layer, b and c are in the same connected component, and are separated from a.
- $\text{f}_4(r)$ to be the number of forests in $T_{r,3}$ such that, in the outer layer, a and c are in the same connected component, and are separated from b.
- $\text{f}_5(r)$ to be the number of forests in $T_{r,3}$ such that, in the outer layer, a, b, and c are all in the same connected component.

Then, we note that

$$\text{f}(r,3) = \text{f}_1(r) + \text{f}_2(r) + \text{f}_3(r) + \text{f}_4(r) + \text{f}_5(r)$$

A careful case analysis was used to determine the following recursive matrix equations, based on which of the outer vertices were in the same connected component of the cycle-free graph. The base cases correspond to the number of forests of each category in the $T_{1,3}$, or $T_{2,3}$, graphs, depending on parity.

$$\begin{bmatrix}\text{f}_1(r)\\ \text{f}_2(r)\\ \text{f}_3(r)\\ \text{f}_4(r)\\ \text{f}_5(r)\end{bmatrix} = \begin{pmatrix} 27 & 15 & 15 & 15 & 7\\ 36 & 23 & 21 & 21 & 11\\ 36 & 21 & 23 & 21 & 11\\ 36 & 21 & 21 & 23 & 11\\ 144 & 95 & 95 & 95 & 53\end{pmatrix} \begin{bmatrix}\text{f}_1(r-1)\\ \text{f}_2(r-1)\\ \text{f}_3(r-1)\\ \text{f}_4(r-1)\\ \text{f}_5(r-1)\end{bmatrix}$$

We go through the details of determining the entries in the first row of the above matrix. The remaining computations are very similar, and exploiting symmetry means many of the entries are repeated.

The first row corresponds to forests that leave the outer vertices in separate connected components. Note that due to symmetry only three cases need be considered:

Case 1: If, before recursively connecting the vertices on the outer layer, the next layer of vertices are in separate connected components. Then, each vertex of the next layer can connect independently to at most one of the vertices on the outer layer. Since each vertex on the next layer is connected to 2 vertices of the outer layer, there are 3 options for each vertex on the next layer. Therefore there are $3^3 = 27$ possible forests of the type $\text{f}_1(r)$ that recursively form from adding a layer to a forest of type $\text{f}_1(r-1)$.

Case 2: If, before recursively connecting the vertices on the outer layer, the next layer of vertices are split among two connected components. Then, the vertex on the next layer lying in its own connected component may connect independently to at most one of the vertices of the outer layer. The two connected vertices on the next layer may connect, as a pair, to at most one of the vertices of the outer layer. Therefore, there are $3 \cdot 5 = 15$ possible forests of the type $\mathrm{f}_1(r)$ that recursively form from adding a layer to a forest of type $\mathrm{f}_i(r-1)$, for $i = 2, 3, 4$.

Case 3: If, before recursively connecting the vertices on the outer layer, the next layer of vertices are in the same connected component. Then at most one of the outer vertices can be connected to the next layers. Any vertex connected to the next layer may only be connected by one edge, as the next layer is already connected. If none of the outer vertices are connected to the next layer, there is only one forest. If one of the outer vertices is connected, there are three choices for which outer vertex is connected, and two choices for which edge to connect, so there are six forests. Therefore, there are 7 forests of type $\mathrm{f}_1(r)$ that recursively form from adding a layer to type $\mathrm{f}_5(r-1)$.

Therefore, we have the first row of the linked recursions:

$$\begin{aligned}\mathrm{f}_1(r) =& 27\mathrm{f}_1(r-1) + 15\mathrm{f}_2(r-1) \\ &+ 15\mathrm{f}_3(r-1) + 15\mathrm{f}_4(r-1) + 7\mathrm{f}_5(r-1)\end{aligned}$$

The matrix recursion above simpifies to the linear recursion, which can be solved exactly.

$$\mathrm{f}(r,3) = 145\mathrm{f}(r-1,3) - 868\mathrm{f}(r-2,3) + 90\mathrm{f}(r-3,3)$$

The asymptotic growth rate of $\mathrm{f}(r,3)$ can then be determined by the largest eigenvalue of the above matrix or, equivalently, by the largest root of $r^3 = 145r^2 - 868r + 90$. This is approximately 138.749, yielding a growth rate of $\Omega(138.749^r)$. □

Since $r \approx \frac{n}{3}$, this yields the asymptotic growth rate in n of $\mathrm{f}(r,3) = \Omega\left(\sqrt[3]{138.749}^n\right) = \Omega(5.17698^n)$.

Corollary 4 *There exist n-vertex planar graphs with $\Omega\left(\sqrt[3]{138.749}^n\right) = \Omega(5.17698^n)$ forests.*

The current upper bound for the number of forests in a planar graph is $n \cdot 6.4948^n$ [4]. The gap between the upper and lower bounds suggest that there may be other graphs that achieve a higher number of forests. Future work will look at computationally determining the value for $T_{r,s}$ with $s > 3$.

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